MA1513 SUPPLEMENTARY NOTES

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1. Week 1: Linear System and Matrix Algebra

Definition 1.1 (Row echelon form (REF))**.** For each non-zero row, the leading entry is to the right of the leading entry of the row above.

E.g. $\begin{pmatrix} 0 & \mathbf{1} & 7 & 2 \\ 0 & 0 & \mathbf{9} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note that the leading entry 9 of the second row is to the right of the leading entry 1 of the first row.

Definition 1.2 (Reduced row echelon form (RREF))**.** A row echelon form is "reduced", if in each of its pivot columns, the leading entry is 1 and all other entries are 0.

E.g. $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$

1.1. **Elementary Row Operations.**

(1) cR_i – multiply the ith row by the constant c
(2) $R_i \leftrightarrow R_j$ – swap the ith and the jth row
(3) $R_i + cR_j$ – add c times of the jth row to the ith row.

1.2. **Gaussian Elimination Summary.** Gaussian Elimination is essentially using the elementary row operations (in any order) to make the matrix to row echelon form.

1.3. **Gauss-Jordan Elimination.** After reaching row echelon form, continue to use elementary row operations to make the matrix to *reduced* row echelon form.

2. Week 2

2.1. **Dot Product.**

- $\mathbf{u} \cdot \mathbf{v} = \|u\|\|v\| \cos\theta$
- $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|u\|\|v\|}$

2.2. **Span.**

$$\begin{aligned} \text{span}\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\} &= \{c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k} \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\} \\ &= \text{set of all linear combinations of } \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}. \end{aligned}$$

2.3. **Subspaces.** $V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

(1) $V = \text{span}\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}$ for some vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$.
(2) V satisfies the *closure properties*:
  (i) for all $\mathbf{u}, \mathbf{v} \in V$, we must have $\mathbf{u} + \mathbf{v} \in V$.
  (ii) for all $\mathbf{u} \in V$ and $c \in \mathbb{R}$, we must have $c\mathbf{u} \in V$.
(3) V is the solution set of a homogeneous system.

(Sufficient to check either one of Condition 1, 2, 3.)

Remark 2.1. For V to be a subspace, zero vector $\mathbf{0}$ must be in V. (Since for $\mathbf{u} \in V$, $0 \in \mathbb{R}$, we have $0\mathbf{u} \in V$.)

2.4. **Linear Independence and Dependence.** $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ are linearly independent if the system

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k} = 0$$

has only the trivial solution, i.e. $c_1 = c_2 = \cdots = c_k = 0$.

If the system has non-trivial solutions, i.e. at least one c_i not zero, then $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ are linearly dependent.

3. Week 3

3.1. **Rotation Matrix (Optional).** The rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates points in the xy-plane counterclockwise through an angle θ about the origin.

For example rotating the vector $(1, 0)$ 45 degrees counterclockwise gives us:

$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

3.2. **Finding Least Squares Solution.** Given $Ax = b$ (inconsistent system), solve

$$A^T Ax = A^T b$$

instead to get a least squares solution of the original equation.

3.3. **Projection.** If we know a least squares solution $\mathbf{u}$ of $A\mathbf{x} = \mathbf{b}$, we can find the projection $\mathbf{p}$ of $\mathbf{b}$ onto the column space of A by

$$\mathbf{p} = A\mathbf{u}.$$

3.4. **Dimension Theorem for Matrices.** If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

(rank(A)=number of pivot columns,
nullity(A)=number of non-pivot columns.)

3.5. Linear Independence and the Wronskian. A set of vector functions $\vec{f_1}(x), \dots, \vec{f_n}(x)$ from $\mathbb{R}$ to $\mathbb{R}^n$ is linearly independent in the interval (α, β) if

$$W[\vec{f_1}(x), \dots, \vec{f_n}(x)] \neq 0$$

for at least one value of x in the interval (α, β).

4. Week 4

4.1. Characteristic Polynomial, $\det(\lambda I - A)$**.**

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff \det(\lambda I - A) = 0 \\ &\iff \lambda \text{ is a root of the characteristic polynomial.} \end{aligned}$$

4.2. Eigenspace. The solution space of $(\lambda I - A)\mathbf{x} = 0$ is called the *eigenspace* of A associated with the eigenvalue λ. The eigenspace is denoted by E_λ.

4.3. Sum/Product of Eigenvalues (Optional).

- The sum of all eigenvalues of A (including repeated eigenvalues) is the same as $\mathrm{Tr}(A)$ (trace of A, i.e. the sum of diagonal elements of A)
- The product of all eigenvalues of A (including repeated eigenvalues) is the same as $\det(A)$.

5. Week 5

5.1. Solutions of Homogeneous Linear System of DE.

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

$$\mathbf{y}(t) = \mathbf{v}e^{rt}$$

where r and $\mathbf{v}$ are eigenvalue and eigenvector for $\mathbf{A}$ respectively.

5.2. Superposition Principle. If $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are two solutions to a homogeneous SDE $\mathbf{y}' = \mathbf{A}\mathbf{y}$, then

$$\mathbf{y} = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

is also a solution for any scalars c_1, c_2.

5.3. Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

5.4. General Solutions (Complex Eigenvalues).

(1) Let $r_1 = a + bi$ be an eigenvalue corresponding to eigenvector $\mathbf{v_1}$. (The eigenvectors are complex conjugates: $\mathbf{v_1}, \mathbf{v_2} = \mathbf{p} \pm \mathbf{q}i$.)

(2) Construct

$$\mathbf{x}_{\mathrm{Re}}(t) = e^{at}(\mathbf{p}\cos bt - \mathbf{q}\sin bt)$$

$$\mathbf{x}_{\mathrm{Im}}(t) = e^{at}(\mathbf{p}\sin bt + \mathbf{q}\cos bt)$$

(3) The general solution is

$$\mathbf{y} = c_1\mathbf{x}_{\mathrm{Re}}(t) + c_2\mathbf{x}_{\mathrm{Im}}(t).$$

References

[1] Prof. Victor Tan's MA1513 Lecture Notes

[2] W.R. Casper's notes on Linear Independence and the Wronskian, https://sites.math.washington.edu/~wcasper/math309_spr16/lectures/lecture4/lecture4.pdf