Tutorial 5

Q1) $F(x,y,z) = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z \end{pmatrix}$

is a conservative vector field.

$$\begin{cases} 1) \quad \bar{F} = \nabla f \quad (\text{potential } f) \\ 2) \quad \int_C F \cdot dr = f(r(b)) - f(r(a)) \\ 3) \quad \int_C \bar{F} \cdot dr = 0 \quad \text{for closed curve } C \end{cases}$$

$r(a) = r(b)$.

Example: Conservative: Gravitational field.

Non-conservative: Friction / Air resistance. (fields with friction)

$$F = \nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \begin{matrix} \leftarrow P \\ \leftarrow Q \\ \leftarrow R \end{matrix}$$

$$\begin{cases} P_y = (f_x)_y = f_{xy} = f_{yx} = Q_x \\ Q_z = f_{yz} = f_{zy} = R_y \\ R_x = f_{zx} = f_{xz} = P_z. \end{cases}$$

(shown) ☆ Continuous second order partial derivatives

$\Rightarrow$ The condition necessary for Mixed Derivative Thm.

$$F \text{ conservative} \implies \begin{cases} P_y = Q_x \\ Q_z = R_y \\ R_x = P_z \end{cases}$$

$$\text{NOT} \begin{cases} P_y = Q_x \\ Q_z = R_y \\ R_x = P_z \end{cases} \implies F \text{ NOT conservative.}$$

$$F(x,y,z) = \begin{pmatrix} 3x-2 \\ z \\ x+y \end{pmatrix} \begin{matrix} \leftarrow P \\ \leftarrow Q \\ \leftarrow R \end{matrix}$$

$P_y = 0$, $Q_x = 0$ ✓

$Q_z = 1$, $R_y = 1$ ✓

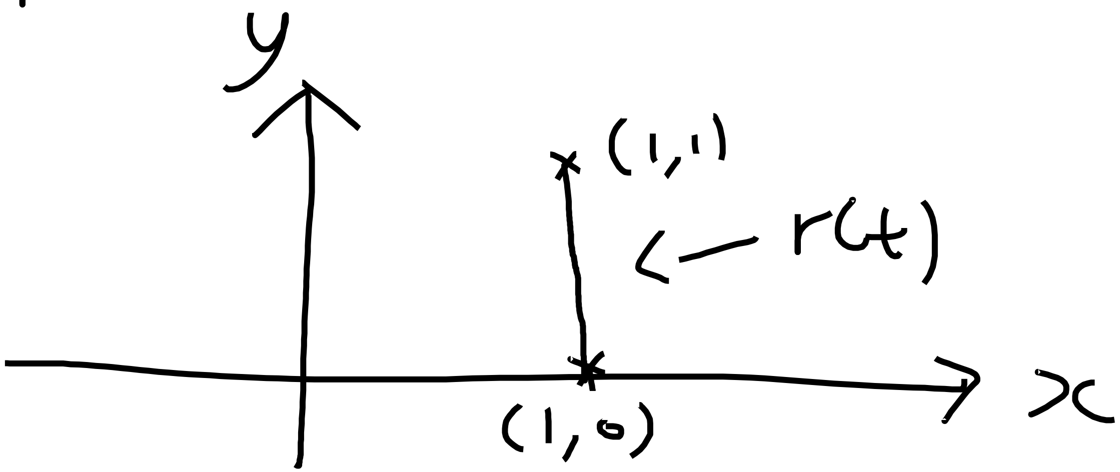
$R_x = 1$, $P_z = 0$ ✗

$\therefore F$ is not conservative.

Q2) $\boxed{\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt.}$

F need not be conservative.
(works for all F)

Step 1) Find $r(t)$.



$$r(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad 0 \leq t \leq 1.$$

$$\underset{\sim}{r} = \underset{\sim}{a} + t\,\underset{\sim}{m}$$

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$F(x,y) = \begin{pmatrix} kx/x^2+y^2 \\ ky/x^2+y^2 \end{pmatrix}$$

$F(r(t))$:
we sub.
$x = 1$
$y = t$ into
$F(x, y)$

$$\int_0^1 F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \begin{pmatrix} k/1+t^2 \\ kt/1+t^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt$$

$$= \int_0^1 \frac{kt}{1+t^2}\, dt = \frac{k}{2}\int_0^1 \frac{2t}{1+t^2}\, dt$$

$$= \frac{k}{2}\left[\ln|1+t^2|\right]_0^1$$

$$= \frac{k}{2}\ln 2.$$

#

☆ Electrical field conservative ✓

We don't know potential f.

Exercise: Calculate potential f ✓

Q3).

$$\operatorname{curl} F = \nabla \times F$$

$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

$$= \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \begin{matrix} \longleftarrow P \\ \longleftarrow Q \\ \longleftarrow R \end{matrix}$$

$$\operatorname{curl}(\nabla f) = \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}$$

$$= \begin{pmatrix} f_{zy} - f_{yz} \\ f_{xz} - f_{zx} \\ f_{yx} - f_{xy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

Mixed Derivative Thm. (Clairaut's Thm.)

Exercise: $\operatorname{div}(\operatorname{curl} F) = 0$.

Q4) $f(x, y, z) = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$.

$$F(x, y, z) = \begin{pmatrix} -kx/(x^2+y^2+z^2)^{3/2} \\ -ky/(x^2+y^2+z^2)^{3/2} \\ \cdots \end{pmatrix}$$

not needed for our question

☆ $\int_C F\, dr = f(r(b)) - f(r(a))$

$r(b)$: endpt.

$r(a)$: start pt.

f is potential fn for F

$\Rightarrow F = \nabla f$.

Let P_1 be (x_1, y_1, z_1)

P_2 be (x_2, y_2, z_2).

$$d_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$d_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}.$$

$$\int_C \underset{\sim}{E} \cdot dr = f(r(b)) - f(r(a))$$

$\vec{F}$

$$= f(P_2) - f(P_1)$$

$$= \frac{k}{\sqrt{x_2^2 + y_2^2 + z_2^2}} - \frac{k}{\sqrt{x_1^2 + y_1^2 + z_1^2}}$$

$$= \frac{k}{d_2} - \frac{k}{d_1}$$

$$= k\left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

#

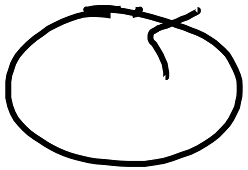
Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

closed curve

Line Integral

Double Integral



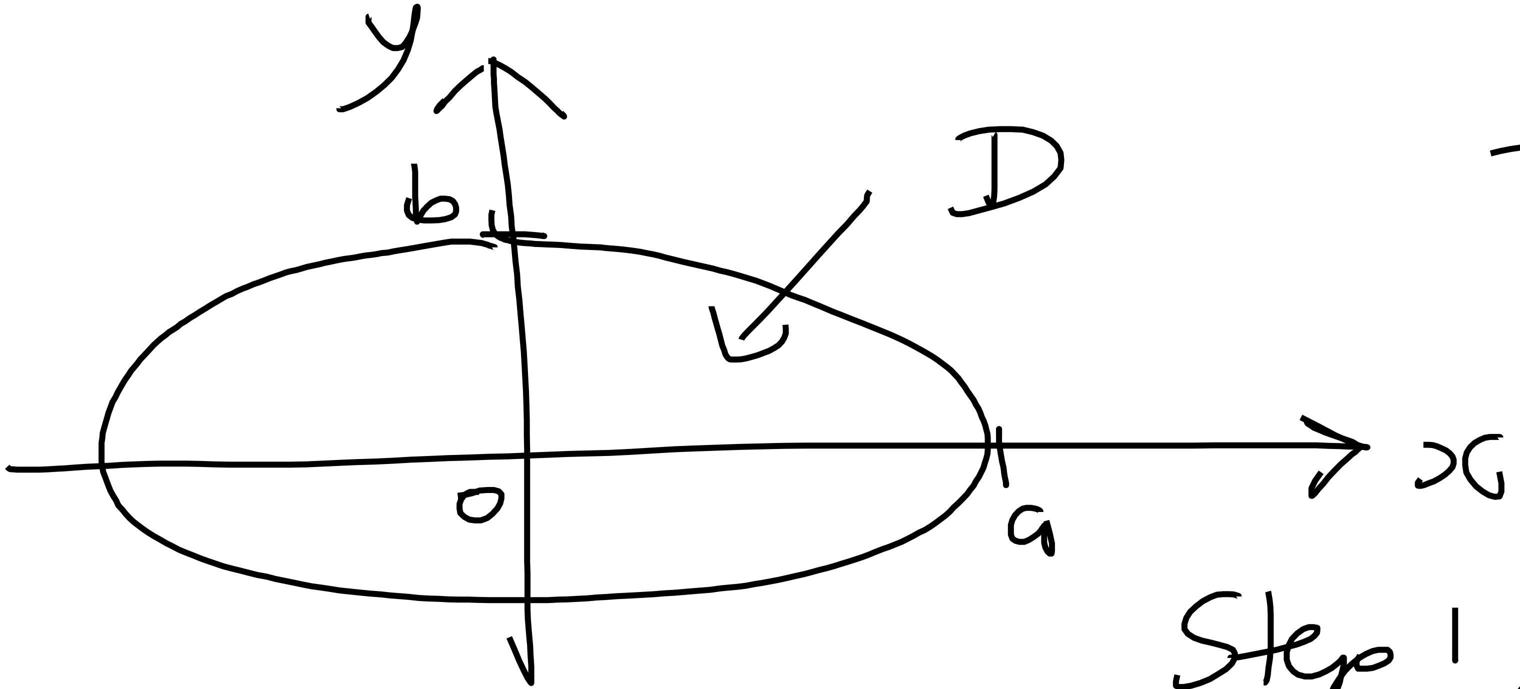
$r(a) = r(b)$

History

George Green (1793 – 1841)

- self-taught (only 1 year primary school)
- father is baker

Application: Area of Ellipse using Green's Theorem



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Area $= \pi ab$.

Step 1) Find $r(t)$.

Parametric Eqn of ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t, \quad 0 \le t \le 2\pi \end{cases}$$

$$\frac{dx}{dt} = -a\sin t$$

$$dx = -a\sin t\, dt$$

$$\boxed{\text{Area of region } D = \iint_D 1 \, dA.}$$

→ Lecture notes
Tutorial (Center of mass)

Idea: Find P, Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$Q = x$
$P = 1$

↓ not as nice

Choose:

$$\begin{cases} P = -\dfrac{y}{2} \\ Q = \dfrac{x}{2} \end{cases}$$

Very nice trick

$$\frac{\partial Q}{\partial x} = \frac{1}{2}, \quad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

$$\text{Area of Ellipse } D = \iint_D 1 \, dA$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

(Green's Thm)

$$= \oint_C P \, dx + Q \, dy$$

$$= \oint_C -\frac{y}{2} \, dx + \frac{x}{2} \, dy$$

$$= \int_0^{2\pi} -\frac{b\sin t}{2} (-a\sin t) \, dt + \frac{a\cos t}{2} (b\cos t) \, dt$$

$$= \int_0^{2\pi} \frac{ab}{2} \left(\sin^2 t + \cos^2 t \right) dt$$

$$= \int_0^{2\pi} \frac{ab}{2} \, dt$$

$$= \frac{ab}{2} \times 2\pi$$

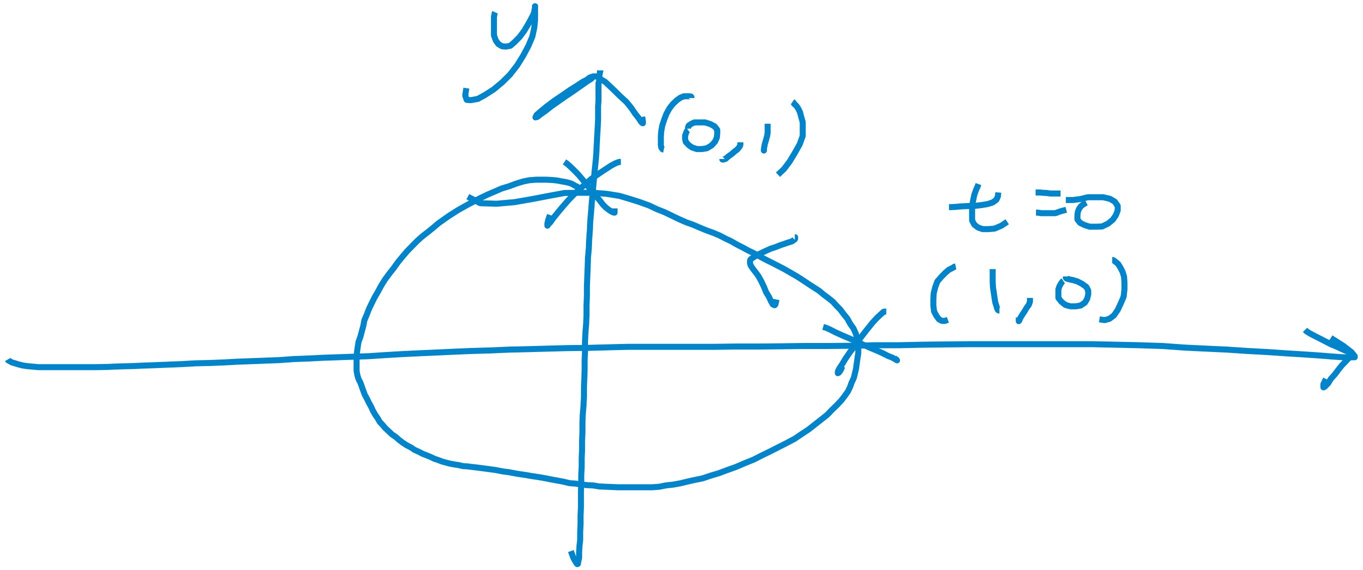
$$= \pi ab$$

#

Polar Coordinates ✓

(is anticlockwise)

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$



At $t=0$,

$x = \cos 0 = 1$

$y = \sin 0 = 0$

At $t = \frac{\pi}{2}$,

$x = \cos \frac{\pi}{2} = 0$

$y = \sin \frac{\pi}{2} = 1$