Tutorial 4

Q1) Let $r(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ be a vector function of constant length.

Constant magnitude

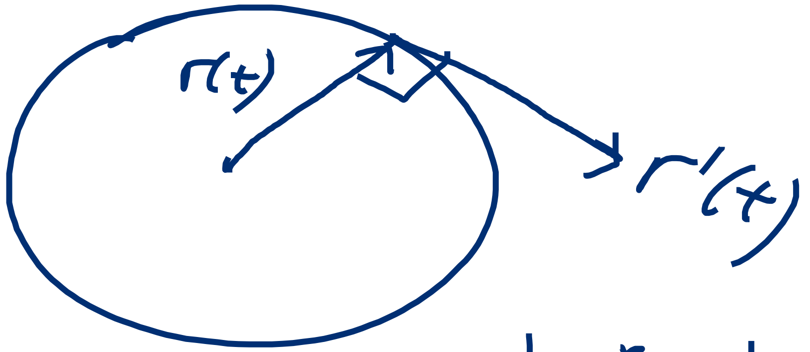
$$\boxed{|r(t)| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2} = C}$$

Show $r(t) \cdot r'(t) = 0$

Application

Position vector $\perp$ velocity vector.

Real-life Application :



- circular motion
- centripetal acceleration

$$|r(t)| = c$$

<u>Proof</u>

$$(x(t))^2 + (y(t))^2 + (z(t))^2 = c^2$$

Diff. w.r.t. t (Implicit)

$$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$$

$$x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = 0$$

Rewrite in dot product notation:

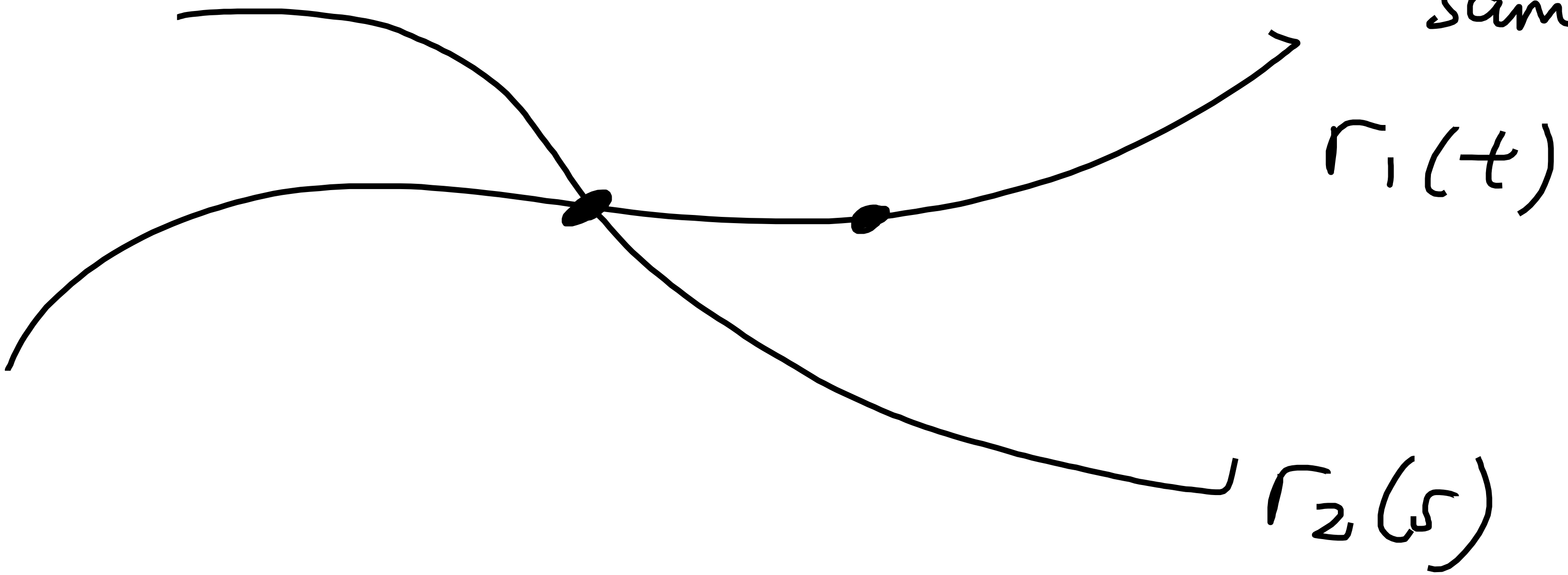
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = 0$$

$$r(t) \cdot r'(t) = 0$$

(shown) #

Q2)

Paths Intersect $\overset{\Longleftarrow}{\not\Longrightarrow}$ Particles Collide [need to be at common pt. at the same time]



(i) $r_1(t) = \begin{pmatrix} 2t+2 \\ t+8 \\ 3t+10 \end{pmatrix}$; $r_2(s) = \begin{pmatrix} s+6 \\ -2s+10 \\ -s+16 \end{pmatrix}$

For intersection questions, the two variables s, t must be different.

$$\begin{cases} 2t+2 = s+6 & \text{—— ①} \\ t+8 = -2s+10 & \text{—— ②} \end{cases}$$

$s=0$, $t=2$.

☆ Need to verify the 3rd Eqn :

$$3t + 10 = 16 \quad \checkmark$$

$$-5 + 16 = 16 \quad \checkmark$$

Yes, the paths meet.

Sub. in $\underline{\underline{s = 0}}$ or $t = 2$ into

$$r_2(0) = \begin{pmatrix} 6 \\ 10 \\ 16 \end{pmatrix} \longleftarrow \text{common point.}$$

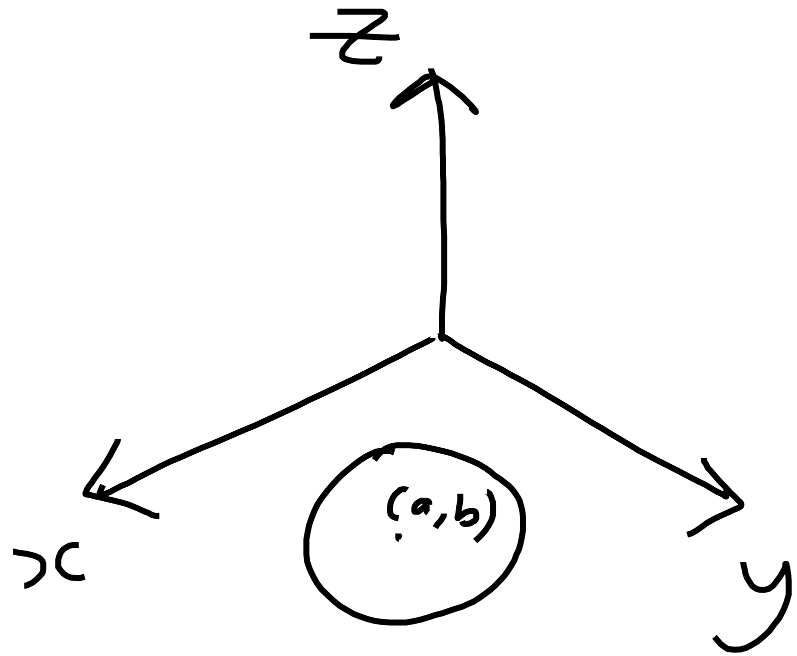
(ii) $\left.\begin{array}{l} s=0 \\ t=2 \end{array}\right\}$ represent time

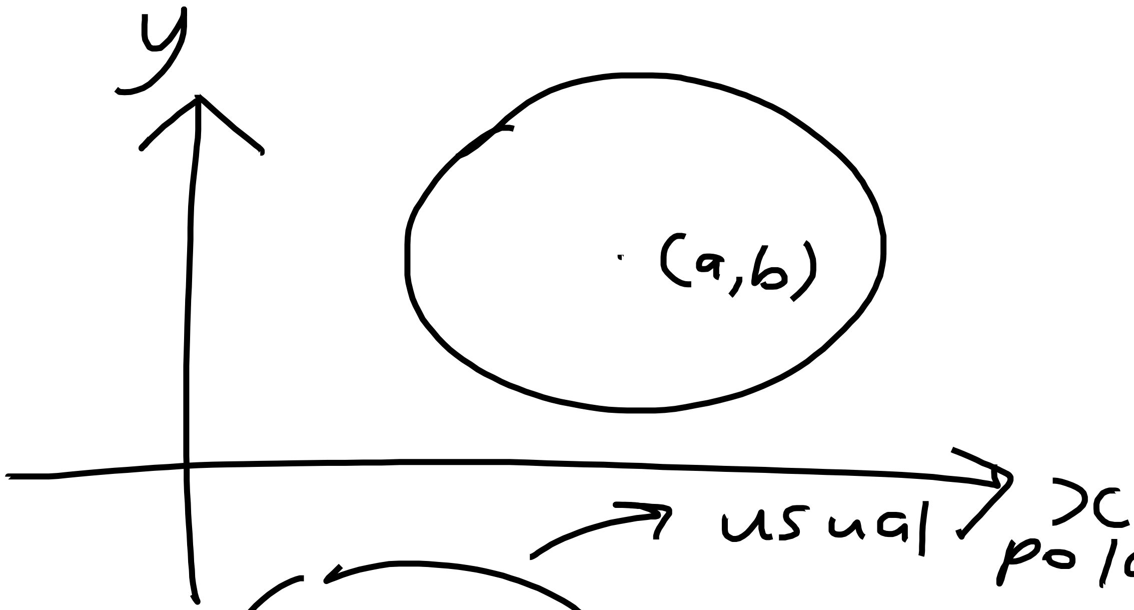
They reach common at different times.

$\therefore$ Do not collide #

Q3) Curvature

Circle of radius R on x-y plane.





usual polar coords formula

$$x = R\cos t + a$$
$$y = R\sin t + b$$
$$z = 0$$

$$r(t) = \begin{pmatrix} R\cos t + a \\ R\sin t + b \\ 0 \end{pmatrix}$$

$$r'(t) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix}$$

$$|r'(t)| = \sqrt{R^2(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{R^2} = R. \qquad \#$$

$$r''(t) = \begin{pmatrix} -R\cos t \\ -R\sin t \\ 0 \end{pmatrix}$$

$$r'(t) \times r''(t) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix} \times \begin{pmatrix} -R\cos t \\ -R\sin t \\ 0 \end{pmatrix}$$

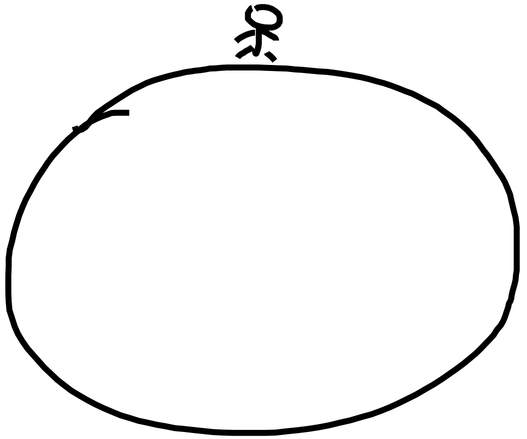
$$= \begin{pmatrix} (R\cos t)(0) - (-R\sin t)(0) \\ -\left[(-R\sin t)(0) - (-R\cos t)(0)\right] \\ (-R\sin t)^2 - (-R\cos t)(R\cos t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R^2 \end{pmatrix}$$

$$|r'(t) \times r''(t)| = R^2$$

$$K(t) = \frac{R^2}{R^3} = \frac{1}{R}.$$

<u>Application</u>

Why the Earth seems flat?



As R is large, $K(t) \approx 0$.

$\iff$ flat

Large radius of circle $\iff$ curvature ≈ 0

(ii) $r(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$

$r'(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$

$|r'(t)| = \sqrt{1 + 4t^2 + 9t^4}$

$r''(t) = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}$

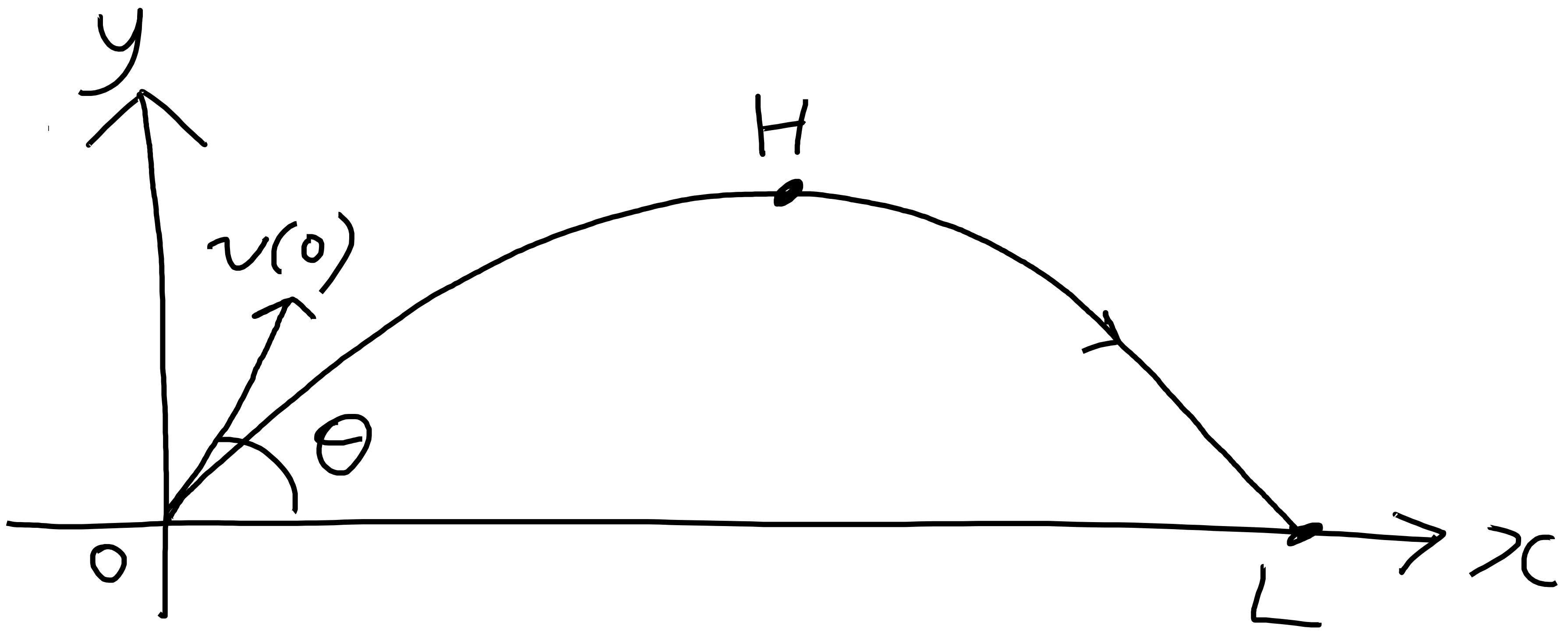
$$r'(t) \times r''(t) = \begin{pmatrix} 6t^2 \\ -6t \\ 2 \end{pmatrix}$$

$$|r'(t) \times r''(t)| = \sqrt{36t^4 + 36t^2 + 4}$$

$$= 2\sqrt{1 + 9t^2 + 9t^4}$$

$$K(t) = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$

Q4)



Galileo proved the shape is parabola

Application

Artillery → Napoleon

Without air resistance

Laplace
Lagrange
Legendre

(i)

$$r(t) \underset{\text{integrate}}{\overset{\text{diff.}}{\rightleftarrows}} v(t) \underset{\text{integrate}}{\overset{\text{diff.}}{\rightleftarrows}} a(t)$$

$$r(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v(0) = \begin{pmatrix} V_0 \cos\theta \\ V_0 \sin\theta \end{pmatrix}$$

$$a(t) = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\Big\downarrow \text{ integrate}$$

$$v(t) = \begin{pmatrix} A \\ -gt + B \end{pmatrix}$$

$$v(0) = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} V_0 \cos\theta \\ V_0 \sin\theta \end{pmatrix}$$

$$v(t) = \begin{pmatrix} V_0 \cos\theta \\ V_0 \sin\theta - gt \end{pmatrix}.$$

$$\Big\downarrow \text{ integrate}$$

$$r(t) = \begin{pmatrix} V_0 t \cos\theta + C \\ V_0 t \sin\theta - \frac{1}{2} g t^2 + D \end{pmatrix}$$

$$r(0) = \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad C = D = 0$$

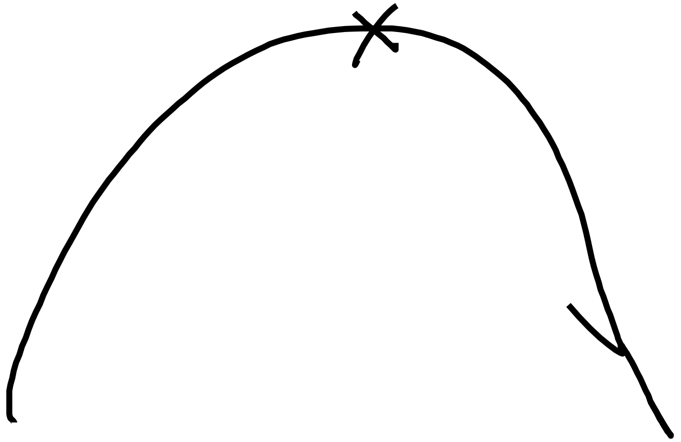
$$r(t) = \begin{pmatrix} (v_0 \cos\theta)t \\ (v_0 \sin\theta)t - \frac{1}{2}gt^2 \end{pmatrix} \begin{array}{l} x(t) \\ y(t) \end{array}$$

(ii) Height

$$r(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

$$Y(t) = (V_0 \sin\theta)\,t - \frac{1}{2} g t^2 .$$

parabola



Find maximum $Y(t)$

$$Y'(t) = V_0 \sin\theta - gt = 0$$

$$t = \frac{V_0 \sin\theta}{g}$$

maximum value

$$Y\left(\frac{V_0 \sin\theta}{g}\right) = (V_0 \sin\theta)\left(\frac{V_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{V_0 \sin\theta}{g}\right)^2 = \frac{(V_0 \sin\theta)^2}{2g}.$$

#

(iii) **Range**

Application

What is best θ for maximum range?

At point L, $Y(t) = 0$.

$$(V_0 \sin\theta)t - \frac{1}{2}gt^2 = 0$$

$$t\left[V_0 \sin\theta - \frac{1}{2}gt\right] = 0$$

$t = 0$ (rejected) or $V_0 \sin\theta - \frac{1}{2}gt = 0$

$$t = \frac{2V_0 \sin\theta}{g} \quad //$$

$$x(t) = (V_0 \cos\theta)\, t$$

$$x\left(\frac{2V_0 \sin\theta}{g}\right) = (V_0 \cos\theta)\left(\frac{2V_0 \sin\theta}{g}\right)$$

$$= \frac{(2\sin\theta\cos\theta)\, V_0^2}{g}$$

$$= \frac{V_0^2 \sin 2\theta}{g} \quad \#$$

$\theta = 45^\circ$ leads to maximum range.

(iv) $$\begin{cases} x = (V_0 \cos\theta)\,t & \text{——— } ① \\ y = (V_0 \sin\theta)\,t - \frac{1}{2} g t^2 & \text{——— } ② \end{cases}$$

Parametric Eqn

Cartesian Eqn (in terms of x, y)

From ①, $$t = \frac{x}{V_0 \cos\theta} \quad \text{——— } ③$$

Sub. ③ into ②.

$$y = (V_0 \sin\theta)\left(\frac{x}{V_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{V_0\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{g\sec^2\theta}{2V_0^2}x^2$$

$\longrightarrow$ parabola.