MA1511

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ALL Discussion Questions

Basic Qns.

In Class Assignment (25min)

Q1(i)

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Show:

$$\frac{1}{x^x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x\ln x)^k$$

(Use substitution)

Replace x with $-x\ln x$

Facts:

$$e^{\ln t} = t \quad (\exp \;\&\; \ln \text{ are inverse})$$

$$e^{-x \ln x} = \sum_{k=0}^{\infty} \frac{1}{k!}(-x \ln x)^k$$

Fact: $n \ln x = \ln x^n$

$$e^{\ln x^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(x \ln x)^k$$

$$x^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x \ln x)^k$$

$$\frac{1}{x^x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x \ln x)^k$$

(shown)

(ii) Integrate both sides.

(ii) $\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(x\ln x)^k\,dx$

☆ Can exchange Integral & Summation
(Valid for MA1511)

$= \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k}{k!}(x\ln x)^k\,dx$

↳ constant wrt x

$$= \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{k!}}_{\text{pull out constant}} \boxed{\int_0^1 (x\ln x)^k\,dx}$$

given

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{\cancel{k!}} (-1)^k \frac{\cancel{k!}}{(k+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}} \overset{\text{replace } k \text{ by } k-1}{=} \sum_{k=1}^{\infty} \frac{1}{k^k} \quad \#$$

Q2) Use $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Show: $\sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$

Idea: $\boxed{\text{Replace}}$ x by $\underline{-x^2}$

$$\sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2}$$

-1

$$\int \sum_{k=0}^{\infty} (-1)^k x^{2k} = \int \frac{1}{1+x^2} \quad \#$$

Hint : $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$

$$\int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx = \int \frac{1}{1+x^2} \, dx$$

$$\sum_{k=0}^{\infty} (-1)^k \int x^{2k} \, dx = \tan^{-1} x + C$$

$$\boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \tan^{-1} x} + C$$

What is C ?

sub: $x=0$

$$0 = 0 + C$$

$$\Rightarrow C = 0$$

(shown)

Q3) Geometric Progression

$X : deposit yearly
(20 times)

1st deposit will earn 20 times
interest

$a_1 = 1.05^{20} X$

2nd deposit : earn 19 years interest

$a_2 = 1.05^{19} X$

$\vdots$

$a_{20} = 1.05^{1} X$

Total sum $= 1.05^1 X + 1.05^2 X + \dots + 1.05^{20} X$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

a : first term

n : number of years

r : common ratio $= 1.05$

$$\frac{1.05X(1.05^{20}-1)}{1.05-1} \geq 200000$$

$$X \geq 5760.5$$

Ans: 5761 (official ans may have problem)

Q4) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{x-1}{2x-3}\right)^k$

Converge for which values of x?

(Radius of convergence)

Idea : Converge iff $\left|\frac{x-1}{2x-3}\right| < 1$

E.g. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{a}{1-r}$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$= 1$$

$\left|\frac{1}{2}\right| < 1$

$|2| > 1$

$2 + 2^2 + 2^3 + 2^4 + \ldots = \infty$

$$\left|\frac{x-1}{2x-3}\right| < 1$$

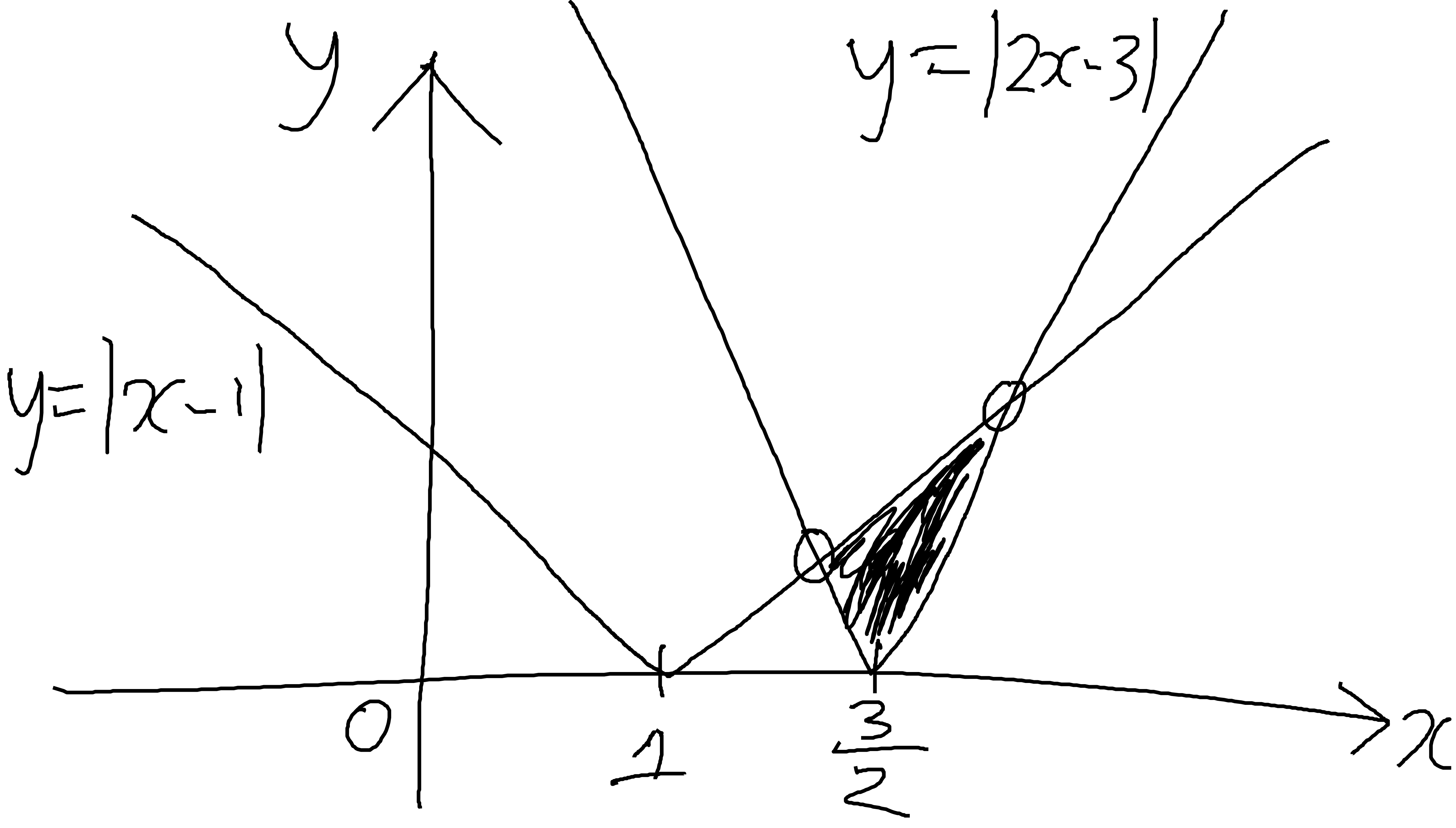
$|x-1|$ graph lower than $|2x-3|$ graph

Graphical way

$$|x-1| < |2x-3|$$

plot

V shape


y
y=|2x-3|
y=|x-1|
x
0
1
3/2

Solve for intersection

$x-1 = 3-2x$ OR $x-1 = 2x-3$

$x = \frac{4}{3}$ OR $x = 2$

$x < \frac{4}{3}$ or $x > 2$.

Q5)

$$\begin{cases} W = T\cos\theta \\ F = T\sin\theta \end{cases}$$

F/W or W/F to eliminate T

Show: $F = \dfrac{Wh}{\sqrt{l^2 - h^2}}$

no T no θ

Idea: Eliminate T & θ

(express in terms of W, h, l)

$$\frac{F}{W} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta$$

$$\tan\theta = \frac{opp'}{adj'} = \frac{h}{\sqrt{l^2 - h^2}}$$

$$F = W\tan\theta = \frac{Wh}{\sqrt{l^2 - h^2}} \quad \#$$

$$F = \frac{Wh}{\sqrt{l^2 - h^2}}$$

$$= \frac{Wh}{\sqrt{l^2}\sqrt{1 - \frac{h^2}{l^2}}}$$

$$= \frac{Wh}{l}\left(1 - \frac{h^2}{l^2}\right)^{-\frac{1}{2}}$$

Binomial i

$(1+x)^{\alpha}$

↳ make it $\underline{1}$

$\frac{h^2}{l^2}$ small

(smaller than $\frac{h}{l}$)

$$= \frac{Wh}{l}\left(1 + \left(-\frac{1}{2}\right)\left(-\frac{h^2}{l^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{h^2}{l^2}\right)^2\right)$$

$$\approx \frac{Wh}{l}\left(1 + \frac{h^2}{2l^2} + \frac{3h^4}{8l^4}\right)$$

Binomial:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Basic Qns:

8) $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$$e^{x} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k}, \quad \text{for all } x \in \mathbb{R}$$

$$e^{-x} = \sum_{k=0}^{\infty} \frac{1}{k!} (-x)^{k} \quad \checkmark$$

$$\frac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$=\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{k!}x^{k}-\sum_{k=0}^{\infty}\frac{1}{k!}(-x)^{k}\right)$$

common factor !

$$=\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\underline{\underline{\left(1-(-1)^{k}\right)}}x^{k}\right)$$

Suggestion: Write a few terms to see the "pattern"

$$= \frac{1}{2}\left(0x^0 + \frac{2}{1!}x^1 + 0x^2 + \frac{2}{3!}x^3 + 0x^4 + \frac{2}{5!}x^5 + \cdots \right)$$

$0! = 1$

Even powers : Zero

Odd powers : $\frac{2}{1!}x^{1} + \frac{2}{3!}x^{3}$

$+ \cdots$

$\boxed{2k+1}$ $\boxed{\frac{2}{(2k+1)!}x^{2k+1}}$

Even : $2k$

$$= \frac{1}{\cancel{2}}\left(\sum_{k=0}^{\infty} \frac{\cancel{2}}{(2k+1)!} x^{2k+1}\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

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In Class Assignment :

10:20 — 10:45 am

Discuss with friend ✓

Ask tutor ✓

No copying ✗