Discussion Problem

Q1)

$$f(x,y) = a(x)\,b(y)$$

$$\int_a^b \int_c^d \boxed{f(x,y)}\, dy\, dx$$

$$= \int_a^b \int_c^d \underbrace{\boxed{a(x)}}_{\text{const. wrt. } y} b(y)\, \underline{dy}\, dx$$

$$= \int_a^b a(x) \underbrace{\boxed{\int_c^d b(y)\, dy}}_{\text{const. (definite integral)}} dx$$

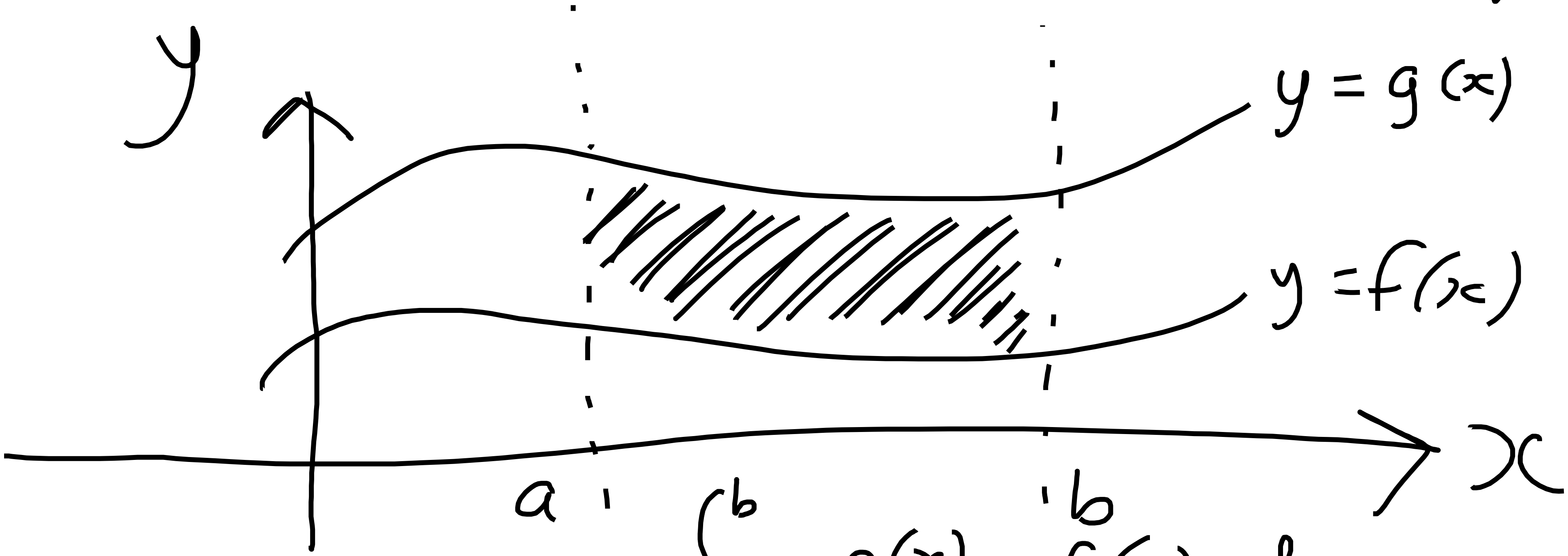
$$= \left(\int_c^d b(y)\, dy \right) \left(\int_a^b a(x)\, dx \right)$$

$\#$

(shown)

Q2) Double integral : Volume under graph

Single integral : Area under graph.



$$\int_a^b g(x) - f(x) \, dx$$

Volume of the solid btw. two surfaces :

$$\iint_D h(x,y) - g(x,y) \, dA$$

(a) $z = 2 - x^2 - y^2$ (Parabolid)

↳ 3D Parabola

$z = 1$

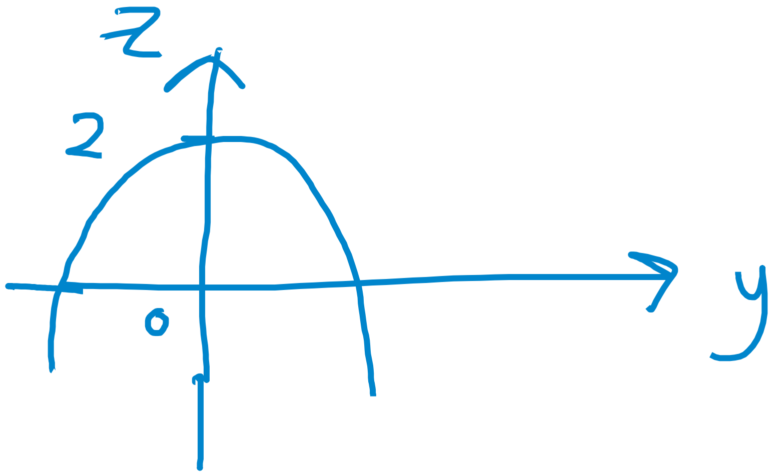
$$z = 2 - x^2 - y^2$$

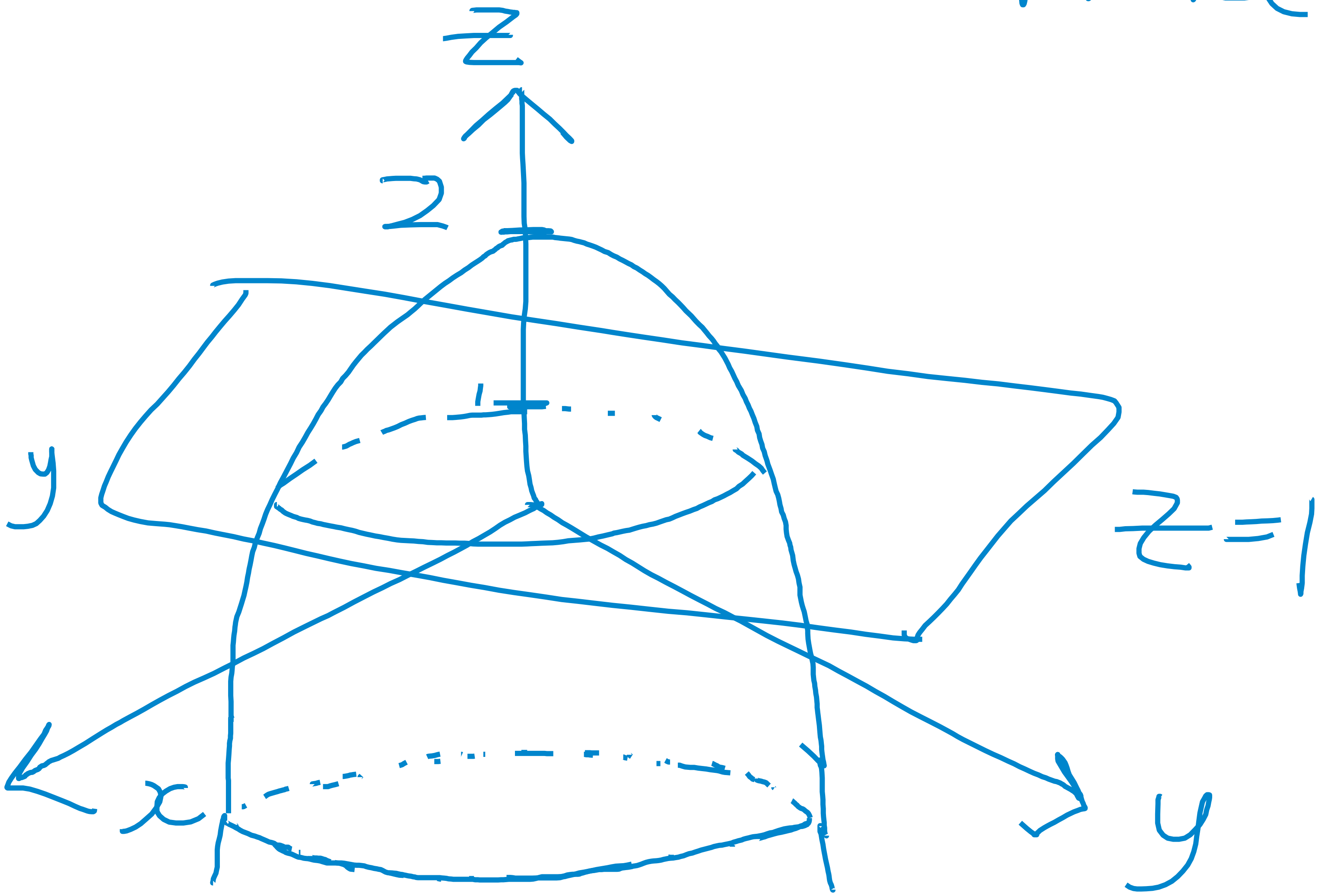
$z = 1$

Plane

Let $x = 0$

$$z = 2 - y^2$$





Step 1) Determine region D.

$$\begin{cases} z = 2 - x^2 - y^2 \\ z = 1 \end{cases}$$

$$2 - x^2 - y^2 = 1$$

$$\boxed{x^2 + y^2 = 1} \quad D$$

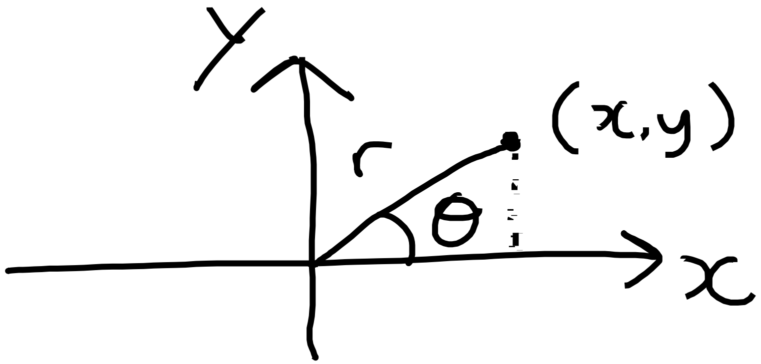
circle radius 1 centered at origin.

$$V = \iint_D (2 - x^2 - y^2) - 1 \; dA$$

$$= \iint_D 1 - x^2 - y^2 \; dA \qquad dA = dx\,dy$$

Cartesian (x, y)

Polar Coordinates (r, θ)

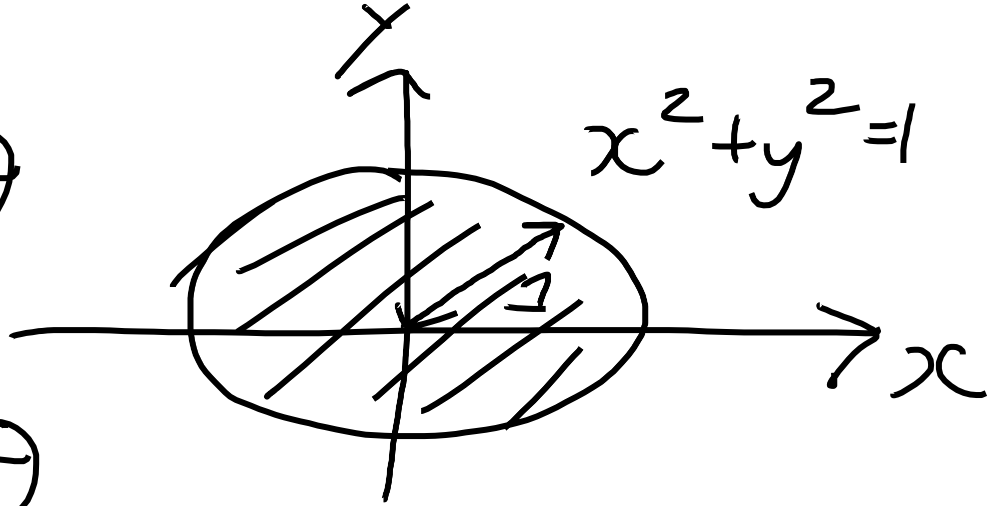


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2 + y^2 \\ dx\,dy = \underline{r}\,dr\,d\theta \end{cases}$$

$$= \iint_D 1 - x^2 - y^2 \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 \, dr\, d\theta$$



$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) \, d\theta$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2} \quad \#$$

$(-\pi$ to $\pi)$ ✓ correct

2b).
$$\begin{cases} z = 2x^2 + y^2 & \text{(Bottom)} \\ z = 27 - x^2 - 2y^2 & \text{(Top)} \end{cases}$$

Step 1) Find domain D.

$$2x^2 + y^2 = 27 - x^2 - 2y^2$$
$$3x^2 + 3y^2 = 27$$
$$x^2 + y^2 = 9$$

circle radius 3

polar coordinates

$$V = \iint_D \text{Top Surface} - \text{Bottom Surface} \, dA$$

$$= \iint_D (27 - x^2 - 2y^2) - (2x^2 + y^2) \, dA$$

$$r^2 = x^2 + y^2$$

$$= \iint_D 27 - 3x^2 - 3y^2 \, dA$$

$$= \int_0^{2\pi} \int_0^3 (27 - 3r^2) \, \underline{\underline{r \, dr \, d\theta}}$$

$$= \dots = \frac{243\pi}{2}$$

#

Q3)

$$m = \iint_R \rho(x,y)\, dA.$$

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

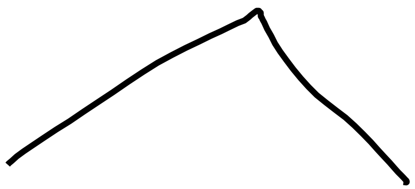
$$\text{mass} = \text{density} \cdot \text{volume}$$

$$\iint_R dA$$

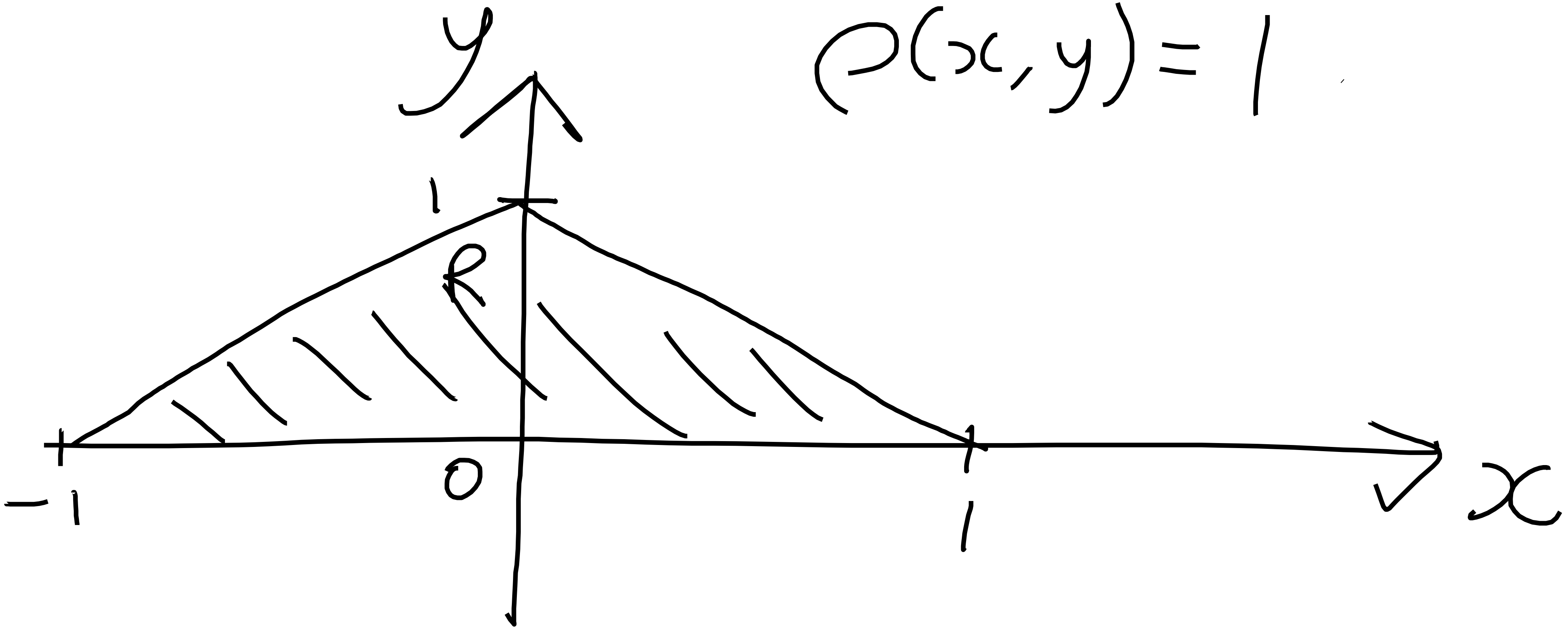
$$\bar{x} = \frac{1}{m} \iint_R x \rho(x,y) \, dA$$

$$\bar{y} = \frac{1}{m} \iint_R y \rho(x,y) \, dA$$

(a) $y = 1 - |x|$



$\rho(x, y) = 1$

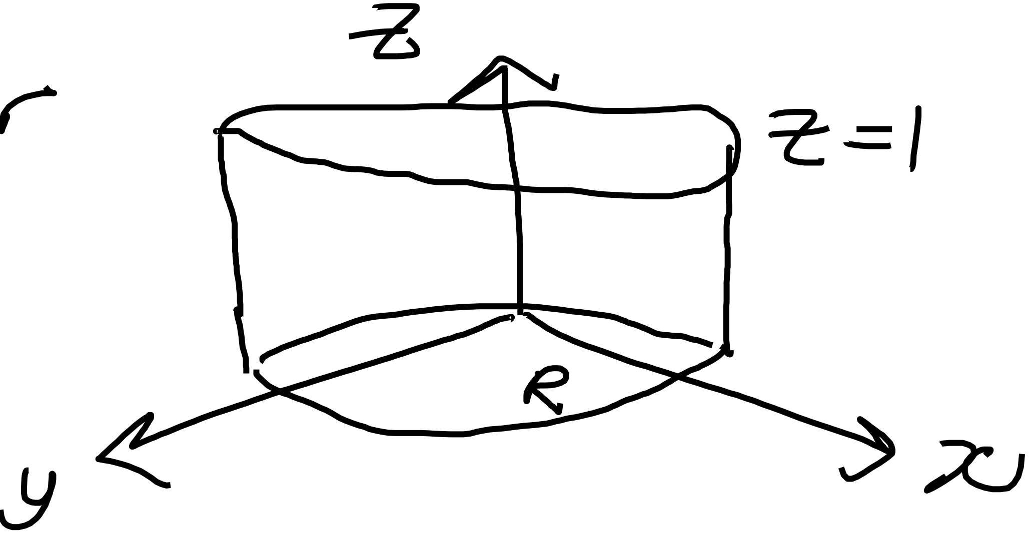


$$m = \iint_R 1 \, dA = \frac{1}{2}(2)(1) = 1$$

Trick : $\boxed{\iint_R 1 \, dA = \text{area of } R}$

Volume under graph $z=1$.

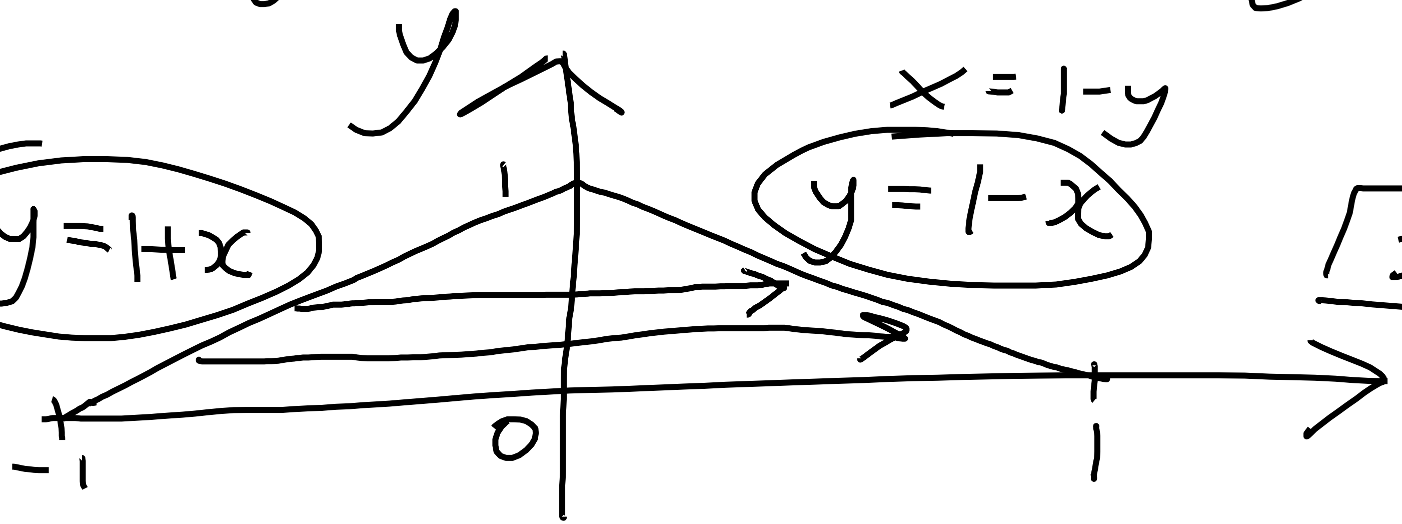
Volume $=$ (area of R)(1)



$$\bar{x} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dA$$

$$= \int_0^1 \int_{y-1}^{1-y} x \; \underline{\underline{dx}}\,dy$$

y ranges from 0 to 1



$x = y-1$

$y = 1+x$

$x = 1-y$

$y = 1-x$

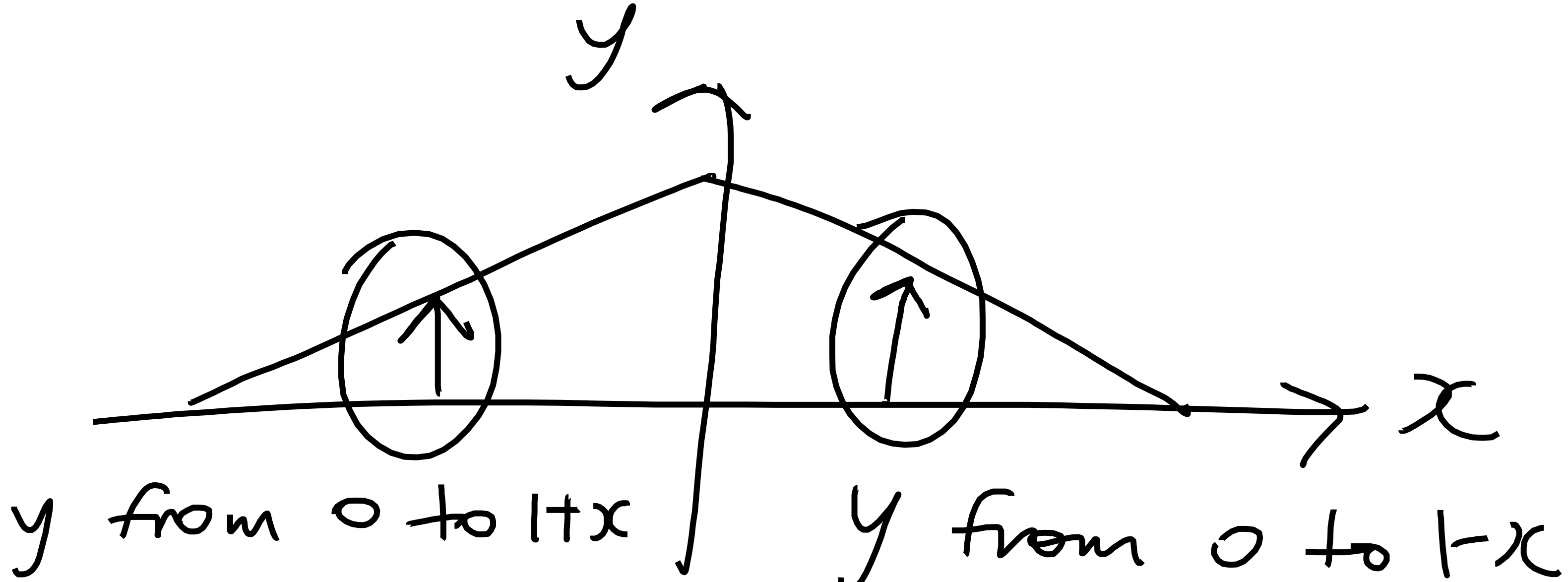
$|x| = 1-y$

$\boxed{x = 1-y}$ or $\boxed{x = y-1}$

$$= \int_0^1 \int_{y-1}^{1-y} x \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} \right]_{y-1}^{1-y} dy$$

$$= 0$$



3(a)

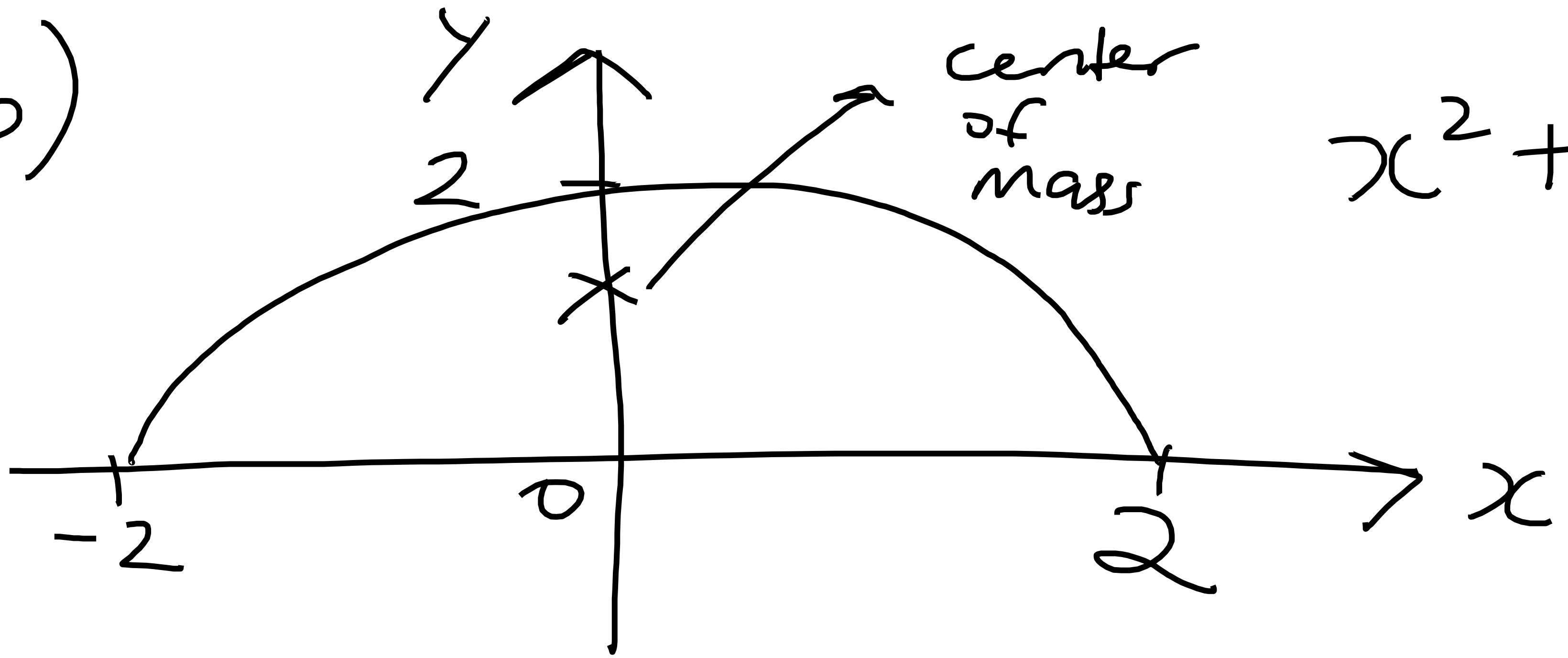
$$\bar{y} = \frac{1}{m}\iint_R y\,\rho(x,y)\,dA$$

$$= \int_0^1 \int_{y-1}^{1-y} y \quad dx\,dy$$

$$= \int_0^1 y\,(1-y-y+1)\,dy$$

$$= \int_0^1 y\,(2-2y)\,dy = \cdots = \frac{1}{3}$$ #

3b)



$x^2+y^2=4$

$\rho(x,y)=1+\frac{y}{2}$

$m=\iint_R\left(1+\frac{y}{2}\right)dA$

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ r^2=x^2+y^2 \\ dA=dx\,dy \\ \quad =r\,dr\,d\theta \end{cases}$$

$$= \int_0^{\pi} \int_0^2 \left(1 + \frac{1}{2} r \sin\theta\right) r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^2 \left(r + \frac{1}{2} r^2 \sin\theta\right) dr \, d\theta$$

$$= \cdots$$

$$= \frac{6\pi + 8}{3}$$

#

$$\bar{x} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dA$$

$$= \frac{1}{m}\int_0^{\pi}\int_0^{2} (r\cos\theta)\left(1+\frac{1}{2}r\sin\theta\right) r\,dr\,d\theta$$

$$= \ldots$$

$$= 0$$

$$\boxed{2\cos\theta\sin\theta = \sin 2\theta}$$

$$\int \sin 2\theta\, d\theta = \frac{-\cos(2\theta)}{2}$$

Intuitively true:

$\rho(\cancel{x}, y) = 1 + \frac{y}{2}$ independent of x.

left & right equally dense

$$\bar{y} = \frac{1}{m}\iint_R y\left(1+\frac{y}{2}\right) dA$$

$$= \frac{3}{6\pi+8}\int_0^{\pi}\int_0^{2} (r\sin\theta)\left(1+\tfrac{1}{2}r\sin\theta\right) r\,dr\,d\theta$$

$$= \frac{3}{6\pi+8}\int_0^{\pi}\int_0^{2} r^2\sin\theta + \tfrac{1}{2}r^3\sin^2\theta \; dr\, d\theta$$

$$= \ldots$$

$$= \frac{3}{6\pi+8}\int_0^{\pi}\left(\frac{8}{3}\sin\theta + \boxed{2\sin^2\theta}\right) d\theta$$

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$$

$$= 2\cos^2\theta - 1$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$=$$

$$= \frac{3\pi + 16}{6\pi + 8} \quad \#$$