MA1511 (Tutorial 2)

Discussion Q 1

$$\boxed{D_u f = \nabla f \cdot \underset{\sim}{u}}$$ unit vector

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$ grad of f

$$D_{(1,0)} f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_x$$

$$D_{(0,1)} f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f_y$$

Q1(i)

$$\boxed{\nabla f(a,b) = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}}$$

$$D_u f = \nabla f \cdot \underset{\sim}{u}$$

Fact: $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

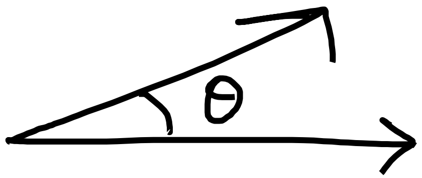
$$= |\nabla f|\,\underbrace{|\underset{\sim}{u}|}_{1}\cos\theta,$$ where θ is the angle btw. ∇f & $\underset{\sim}{u}$.

$$= |\nabla f|\cos\theta$$

Fact: $-1 \le \cos\theta \le 1$

$$\therefore \; -|\nabla f| \le D_u f \le |\nabla f|$$

(i) Max. value occurs when $\cos\theta = 1 \Rightarrow \theta = 0$

$\Rightarrow$ 

$\underset{\sim}{u}$ & ∇f same direction

$$\text{Max. value} = |\nabla f| = \sqrt{f_x^2(a,b) + f_y^2(a,b)}$$

Fact

$$\underset{\sim}{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

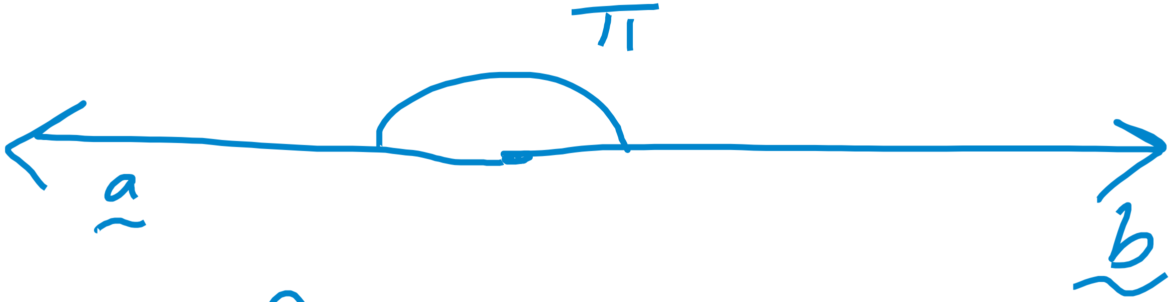
$$|\underset{\sim}{v}| = \sqrt{x^2 + y^2}$$ (Magnitude of vector)

$$\underset{\sim}{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$|\underset{\sim}{u}| = \sqrt{x^2 + y^2 + z^2}$$

(ii) $\cos\theta = -1$

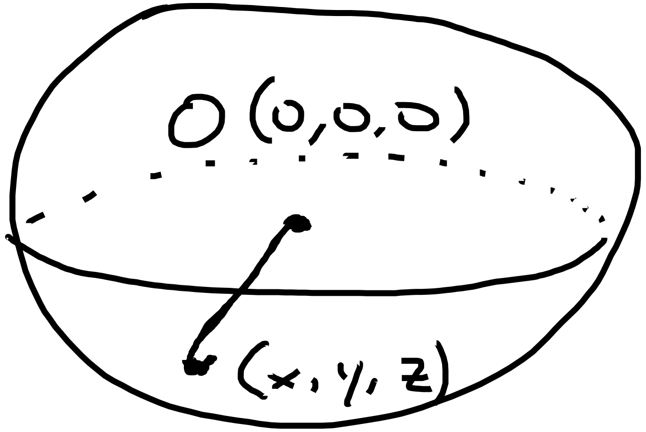
$\Rightarrow \theta = \pi$



$\underset{\sim}{u}$ & ∇f Opposite direction

Min : $-|\nabla f| = -\sqrt{f_x^2(a,b) + f_y^2(a,b)}$ #

Q2) Celsius



Fact :

Distance btw. point & Origin

$= \sqrt{x^2 + y^2 + z^2}$

Let temperature at point (x, y, z) be $T(x, y, z)$.

$$T = \frac{k}{\sqrt{x^2+y^2+z^2}}$$ ← Step 1 : find k

Given : At point (1, 2, 2), $T = 120$

$$120 = \frac{k}{\sqrt{1^2+2^2+2^2}} = \frac{k}{\sqrt{9}}$$

$$k = 360$$ //

$$T(x,y,z) = \frac{360}{\sqrt{x^2+y^2+z^2}}.$$

$$= 360\left(\underline{\underline{x^2+y^2+z^2}}\right)^{-\frac{1}{2}}$$

$$T_x = 360\cdot\left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

diff. w.r.t. x (treat y & z as constants)

$$\times\ (2x)$$

$$= \frac{-360x}{(x^2+y^2+z^2)^{3/2}}.$$

$$T_y = \frac{-360y}{(x^2+y^2+z^2)^{3/2}}$$

$$T_z = \frac{-360z}{(x^2+y^2+z^2)^{3/2}}$$

(i)

$$D_u f = \nabla f \cdot \underset{\sim}{u}$$

$$D_u T = \nabla T \cdot \underset{\sim}{u}$$

$$\nabla T = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix}$$

$$(1,2,2) \longrightarrow (2,1,3)$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

not unit vector

$$\underset{\sim}{u} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

divide by magnitude of # $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

At $(1, 2, 2)$, sub. in $x = 1$
$y = 2$
$z = 2$

$$\Rightarrow T_x = -\frac{40}{3}$$

$$T_y = -\frac{80}{3}$$

$$T_z = -\frac{80}{3}$$

$$D_u T(1,2,2) = \nabla T \cdot \underset{\sim}{u}$$

$$= \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \cdot \underset{\sim}{u}$$

$$= \begin{pmatrix} -40/3 \\ -80/3 \\ -80/3 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= -\frac{40}{3\sqrt{3}}$$

(b) Related to Q1.

By Q1, the most <u>negative</u> (decrease) $D_{u}T$ is when $\underset{\sim}{u}$ is opposite direction with ∇T.

$\Rightarrow$ $\underset{\sim}{u}$ is in direction of $-\nabla T$

Let $\overrightarrow{OP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$-\nabla T(a,b,c) = -\begin{pmatrix} T_x(a,b,c) \\ T_y(a,b,c) \\ T_z(a,b,c) \end{pmatrix}$$

$$= -\begin{pmatrix} \dfrac{-360a}{(a^2+b^2+c^2)^{3/2}} \\ \dfrac{-360b}{(a^2+b^2+c^2)^{3/2}} \\ \dfrac{-360c}{(a^2+b^2+c^2)^{3/2}} \end{pmatrix}$$

$$= \frac{360}{(a^2+b^2+c^2)^{3/2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \frac{360}{(a^2+b^2+c^2)^{3/2}} \overrightarrow{OP}$$

(shown)

in direction of $\overrightarrow{OP}$

Q3). $$\boxed{\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}}$$

(Heat equation) u_t is multiple of u_{xx}

Partial Diff. Eqn. (PDE)

Verify that

$$u(x,t) = e^{-pt}\left(A\sin(qx) + B\cos(qx)\right)$$

is a solution

$$u_t = -p e^{-pt} \underbrace{\left(A\sin(qx) + B\cos(qx) \right)}_{\text{const wrt. } t}$$

$$u_x = e^{-pt}\left(Aq\cos(qx) - Bq\sin(qx) \right)$$

$$u_{xx} = \frac{\partial}{\partial x}(u_x)$$

$$= e^{-pt}\left(-Aq^2\sin(qx) - Bq^2\cos(qx) \right)$$

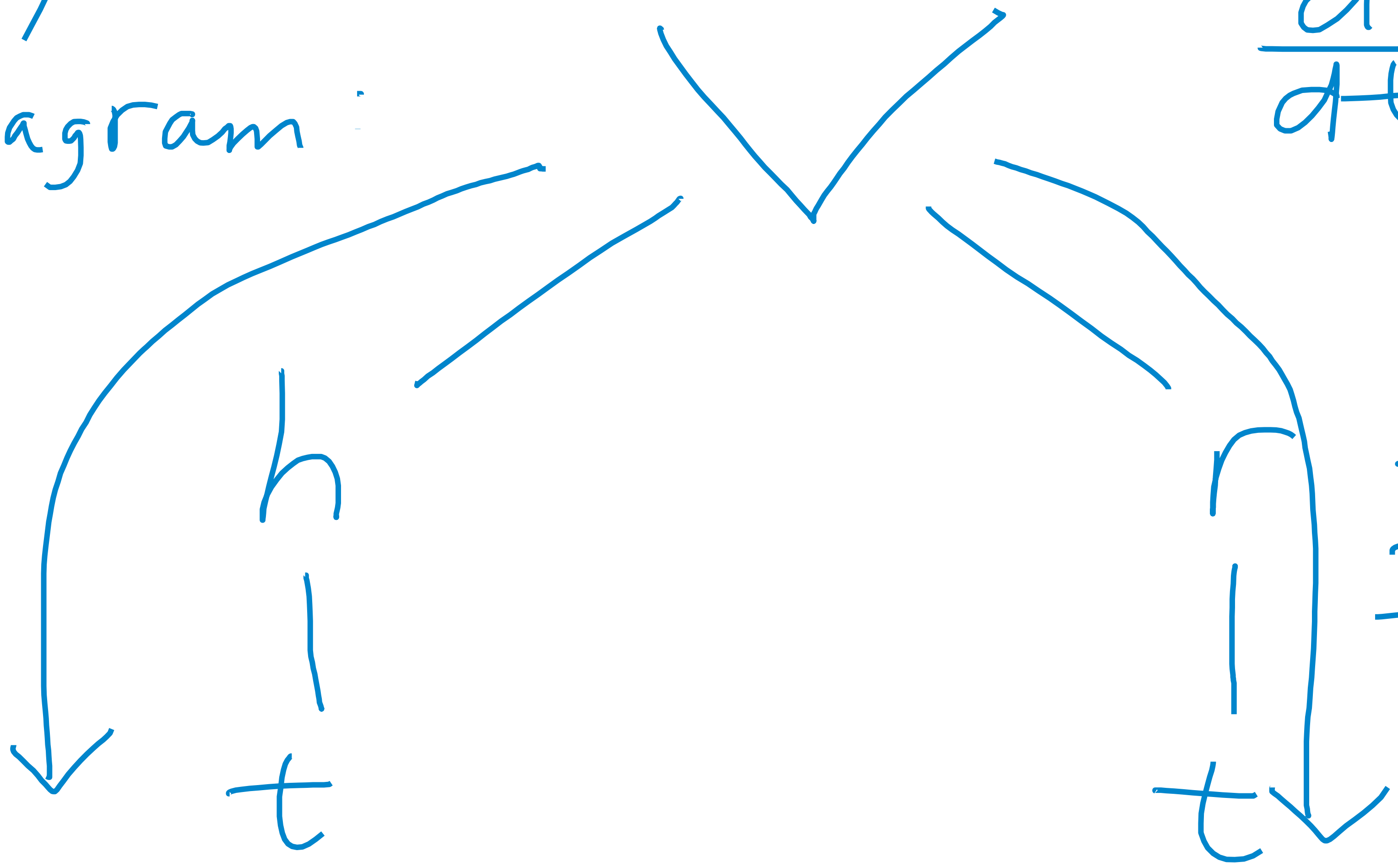
$$= -q^2 e^{-pt}\left(A\sin(qx) + B\cos(qx) \right)$$

$$u_t = \frac{p}{q^2} u_{xx}$$

$$\Rightarrow \quad c = \frac{p}{q^2}$$

Q4) Multivariable Chain Rule

Tree Diagram:



$$\frac{dV}{dt} = \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} + \frac{\partial V}{\partial r} \cdot \frac{dr}{dt}$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial h}\frac{dh}{dt} + \frac{\partial V}{\partial r}\frac{dr}{dt}$$

$$= (\pi r^2)\frac{dh}{dt} + (2\pi rh)\frac{dr}{dt}$$

$$= \pi(15)^2(-3) + 2\pi(15)(40)(-1)$$

$$= -1875\pi$$

Ans: decreasing at 1875π cm^3/h.

Q5) $f_{xy} = f_{yx}$

Clairaut's Thm / Mixed Derivative Thm.

☆ Always valid in MA1511

$$f(x,y) = x^2 + e^{2y}\ln x + \frac{2y^y}{\ln y + 3e^{4y}}$$

$\leftarrow$ hard to diff. w.r.t. y

$$f_{yx} = f_{xy} = (f_x)_y$$

$$= \left(2x + e^{2y}\frac{1}{x} + 0\right)_y$$

$$= 0 + \frac{2}{x}e^{2y} = \frac{2e^{2y}}{x} \quad \#$$

Q6) Summary of Lagrange Multiplier

☆ $$\boxed{\nabla f = \lambda \nabla g}$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

f : function to optimize (max/min)

g : constraint (condition)

$g(x, y, z) = \text{const}$

$$A = \sqrt{s\boxed{(s-a)(s-b)(s-c)}}$$

maximize this

where $s = \frac{1}{2}(a+b+c)$

semiperimeter

☆ "fixed perimeter"

Variables : a, b, c

s : constant.

Maximise A

$\iff$ Maximise

☆ $f(a,b,c) = (s-a)(s-b)(s-c)$

$\boxed{\text{Shortcut}}$

Idea: largest $\sqrt{f(x)} \iff$ largest $f(x)$

Constraint:

$$\boxed{g(a,b,c) = a+b+c = 2s}$$

$$\boxed{\nabla f = \lambda \nabla g}$$

$$f_{\cancel{x}a} = \lambda g_{\cancel{x}a}$$

$$f_{\cancel{y}b} = \lambda g_{\cancel{y}b}$$

$$f_{\cancel{z}c} = \lambda g_{\cancel{z}c}$$

$$f_a = -(s-b)(s-c)$$

$$g_a = 1+0+0 = 1$$

$$-(s-b)(s-c) = \lambda \quad \text{——} \quad (1)$$

$$f_b = -(s-a)(s-c)$$

$$g_b = 1$$

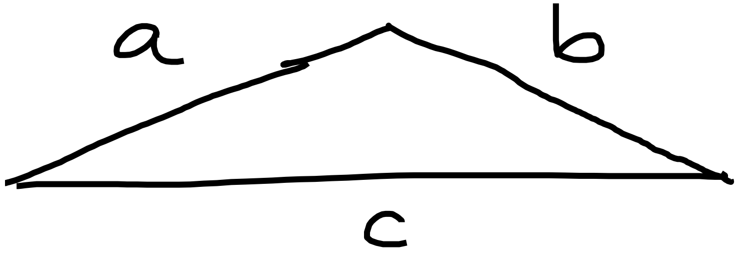
$$-(s-a)(s-c) = \lambda \text{ ——— } ②$$

$$-(s-a)(s-b) = \lambda \text{ ——— } ③$$

$$① = ②, \quad (s-b)(s-c) = (s-a)(s-c)$$

$$\Longrightarrow \overset{(\text{rej.})}{s = c} \text{ or } a = b \checkmark$$

If $s = c \Longrightarrow$



$$\frac{1}{2}(a+b+c) = c$$

$$\frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}c$$

contradiction

$$a+b = c \quad (\Longrightarrow\Longleftarrow)$$

Triangle Inequality:

Sum of two sides of $\triangle$ is greater than 3^{rd} side

2 = 3

$$(s-a)(s-c) = (s-a)(s-b)$$

$\Rightarrow$ $s=a$ or $b=c$ ✓

(rejected)

$a=b$ & $b=c$ ✓

$a=b=c$ (Equilateral)