Q1) $f(x,y) = a(x)\,b(y)$.

e.g. $f(x,y) = x^2 \sin(y)$ ✓

$f(x,y) = (x+y)^{1/3}$ ✗

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_a^b \int_c^d a(x)\,b(y)\,dy\,dx$$

constant wrt. y

$$= \int_a^b a(x) \boxed{\int_c^d b(y)\, dy}\, dx$$

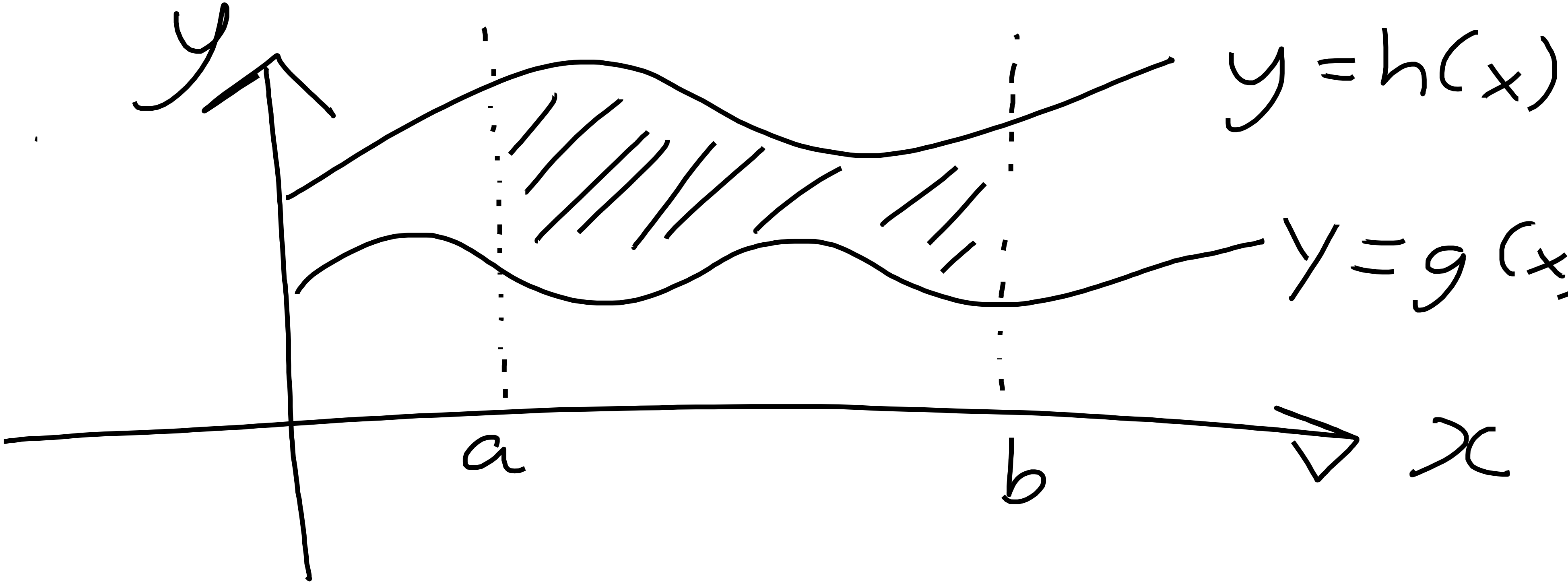
constant

definite integral

$$= \left(\int_c^d b(y)\, dy \right) \left(\int_a^b a(x)\, dx \right)$$

(shown)

Q2)



$$\int_a^b h(x) - g(x) \, dx$$

Area of shaded region

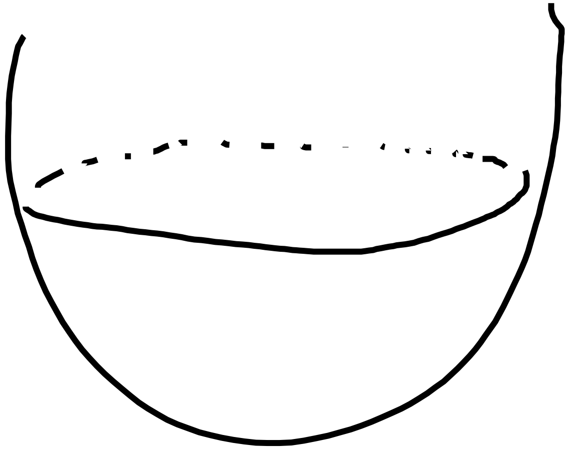
Volume of solid btw 2 surfaces

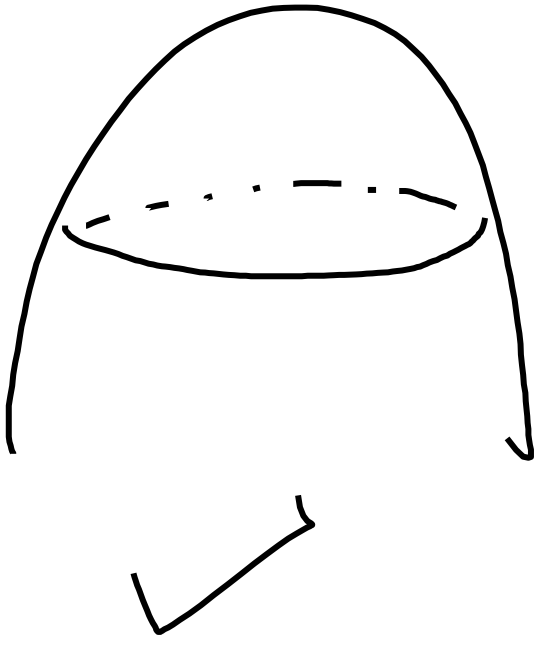
$$\iint_D \underbrace{h(x,y)}_{\text{Top surface}} - \underbrace{g(x,y)}_{\text{Bottom surface}} \, dA$$

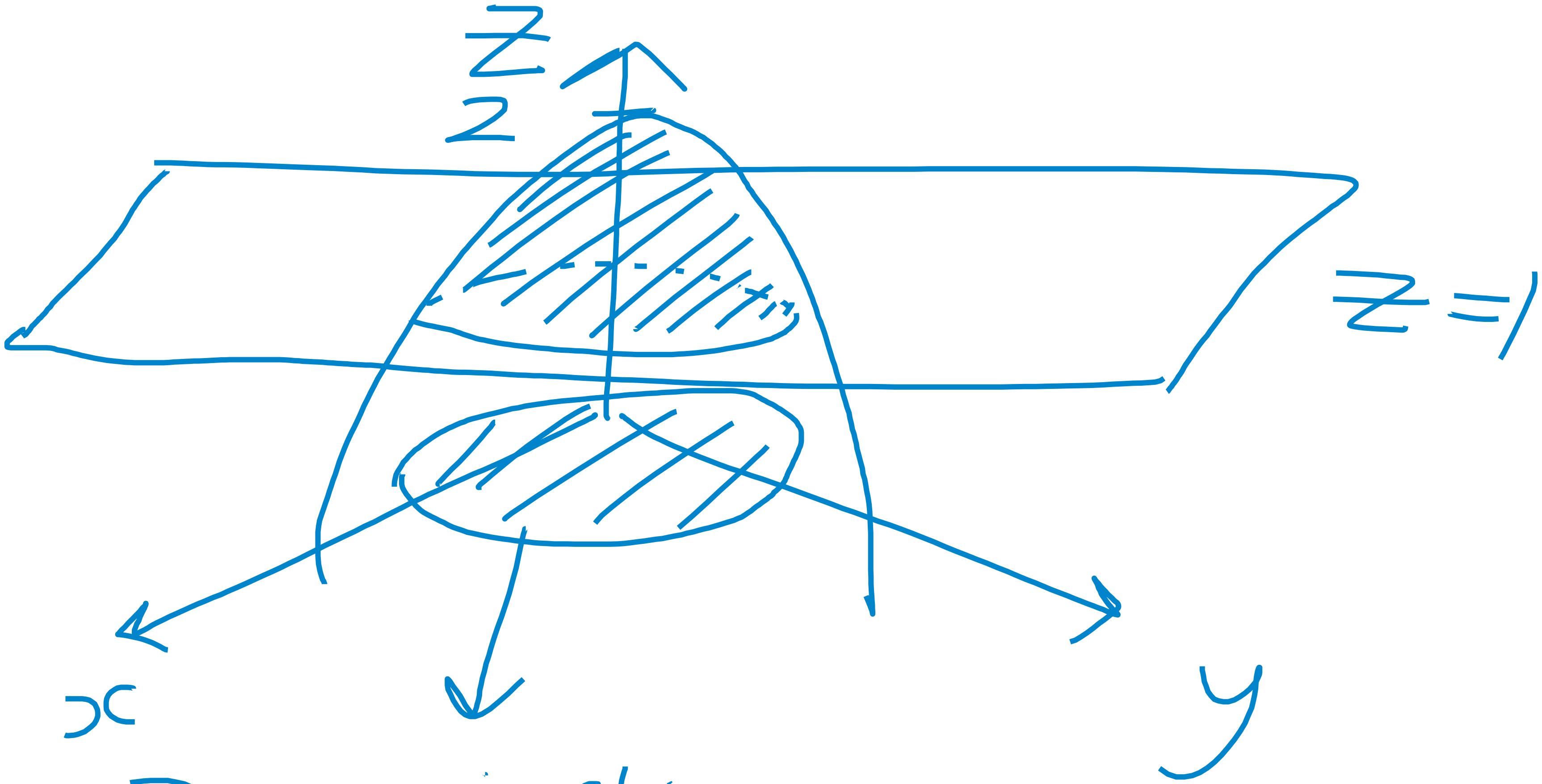
D → domain of integration.

Q2a)

$$\begin{cases} z = 2 - x^2 - y^2 & \text{(paraboloid)} \\ z = 1 & \text{(plane)} \end{cases}$$







D. projection on x-y plane

Step 1) Find D.

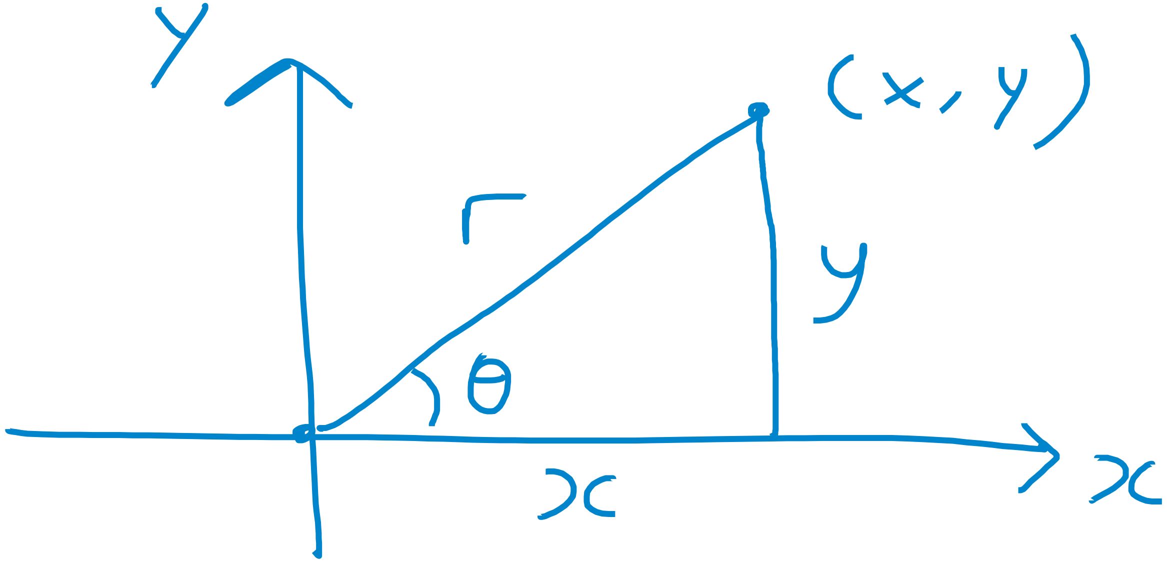
$$2 - x^2 - y^2 = 1$$

$$x^2 + y^2 = 1$$

(circle of radius 1)

$$D = \{(x, y) \cdot x^2 + y^2 \leq 1\}$$

circle $\Rightarrow$ polar coordinates.



$$\cos\theta = \frac{x}{r}$$

4 Impt Formulae :

1. $x = r\cos\theta$
2. $y = r\sin\theta$
3. $x^2 + y^2 = r^2$ (Pythagoras')
4. $dA = dx\,dy = \underline{\underline{r}}\,dr\,d\theta$

$$V = \iint_D \underset{\text{surface}}{\text{Top}} - \underset{\text{surface}}{\text{Bottom}} \, dA$$

$$= \iint_D (2 - x^2 - y^2) - 1 \quad dA$$

$$= \iint_D 1 - x^2 - y^2 \quad dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) \; r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta$$

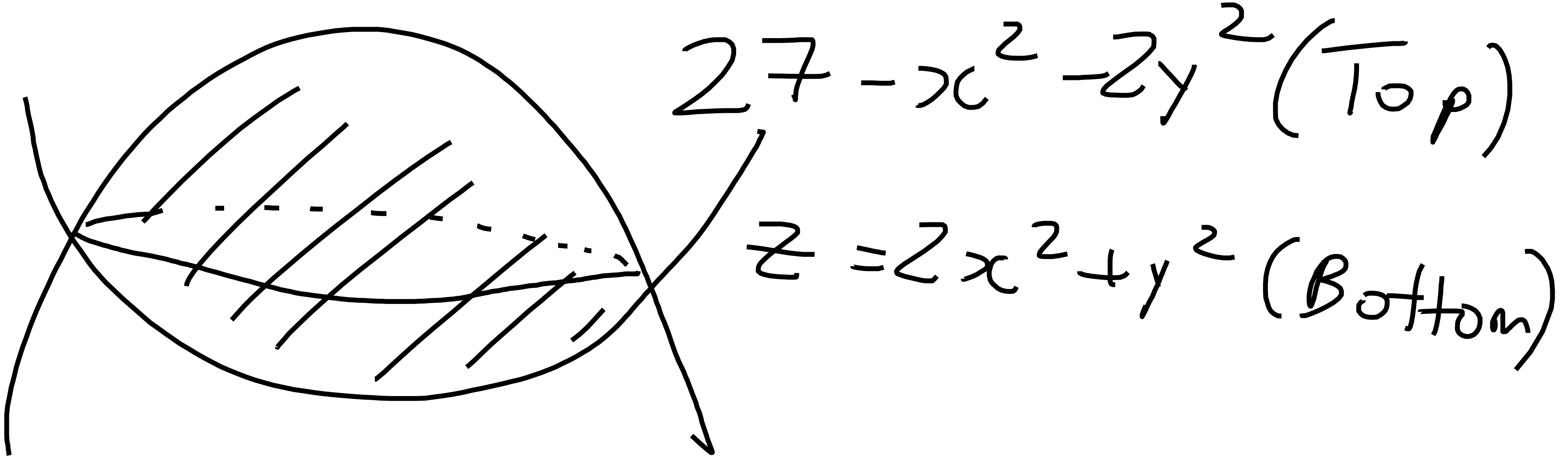
$$= \int_0^{2\pi} \frac{1}{4} \, d\theta$$

$$= \frac{\pi}{2} \quad \#$$

Q2b)

$$z = 2x^2 + y^2$$

$$z = 27 - x^2 - 2y^2$$



Step 1) Find D.

$$2x^2 + y^2 = 27 - x^2 - 2y^2$$

$$3x^2 + 3y^2 = 27$$

$$x^2 + y^2 = 9$$

circle of radius 3.

$$V = \iint_D \text{Top surface} - \text{Bottom surface} \; dA$$

$$= \iint_D (27 - x^2 - 2y^2) - (2x^2 + y^2) \, dA$$

$$= \iint_D 27 - 3x^2 - 3y^2 \quad dA$$

$$= \int_0^{2\pi} \int_0^3 (27 - 3r^2) r \, dr \, d\theta$$

$=$

$$= \frac{243\pi}{2} \quad \#$$

Q3) Center of mass

$$m = \iint_R \rho(x,y)\, dA$$

mass = density × volume

$$\bar{x} = \frac{1}{m} \iint_R x\rho(x,y)\, dA$$

$$\bar{y} = \frac{1}{m}\iint_R y\,\rho(x,y)\,dA.$$

(a) $y = 1 - |x|$

$\rho(x,y) = 1$

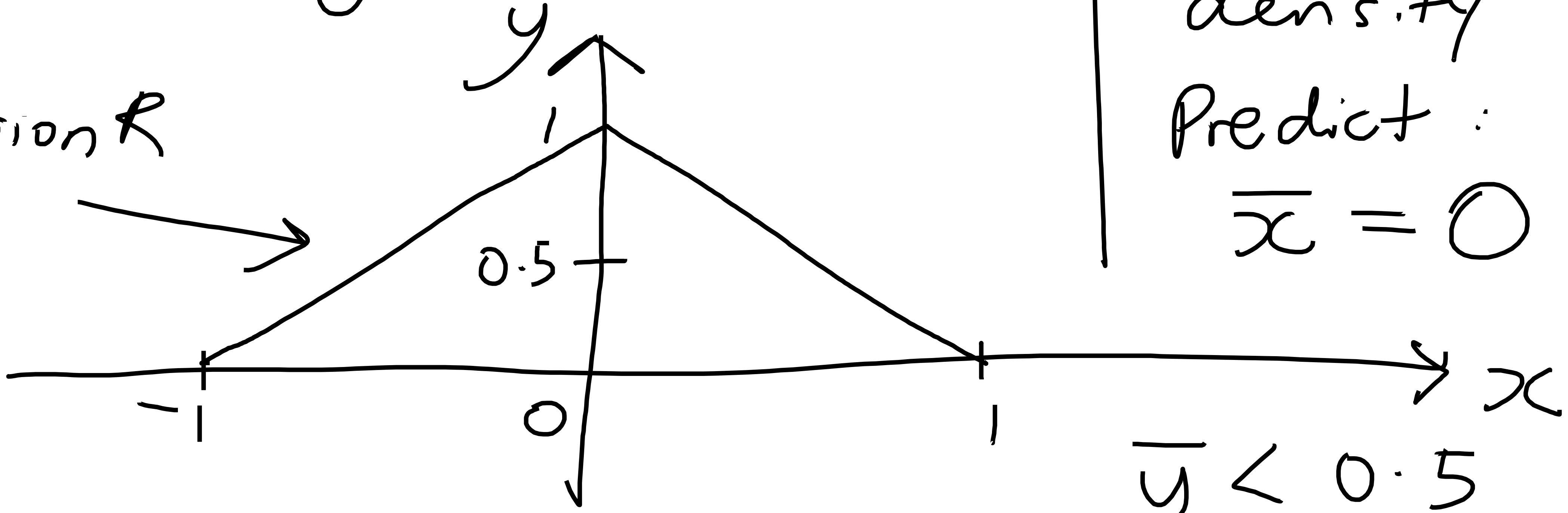
uniform density

Predict:

$\bar{x} = 0$

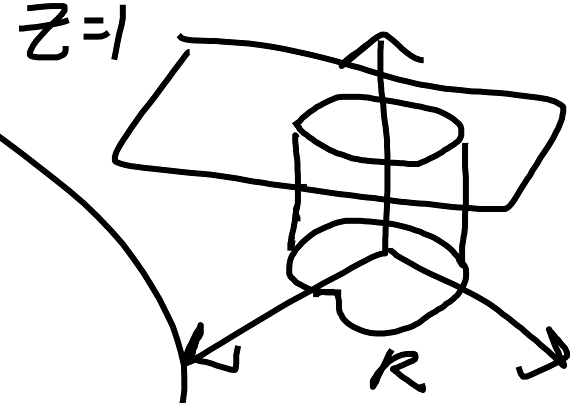
$\bar{y} < 0.5$

region R

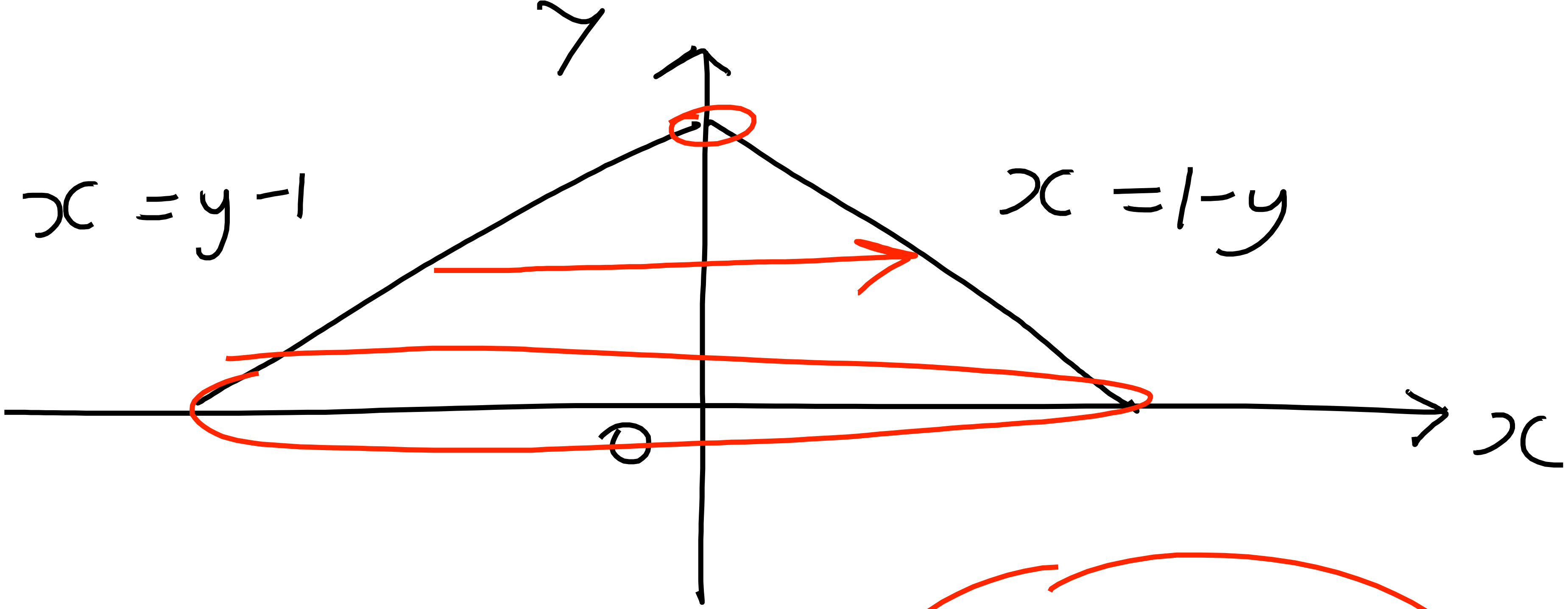


$$m = \iint_R 1 \, dA = \frac{1}{2} \times 2 \times 1$$
$$= 1 \quad \#$$

Trick: $\iint_R 1 \, dA$ = area of region R



Double integral = Volume under surface ($z=1$)
= Base area × height
= area of R × 1



Type 1 or Type 2

Horizontal lines

$$\bar{x} = \frac{1}{m}\int_0^1 \int_{y-1}^{1-y} x \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} \right]_{y-1}^{1-y} dy$$

$$= \int_0^1 0 \, dy$$

$$= 0$$

#

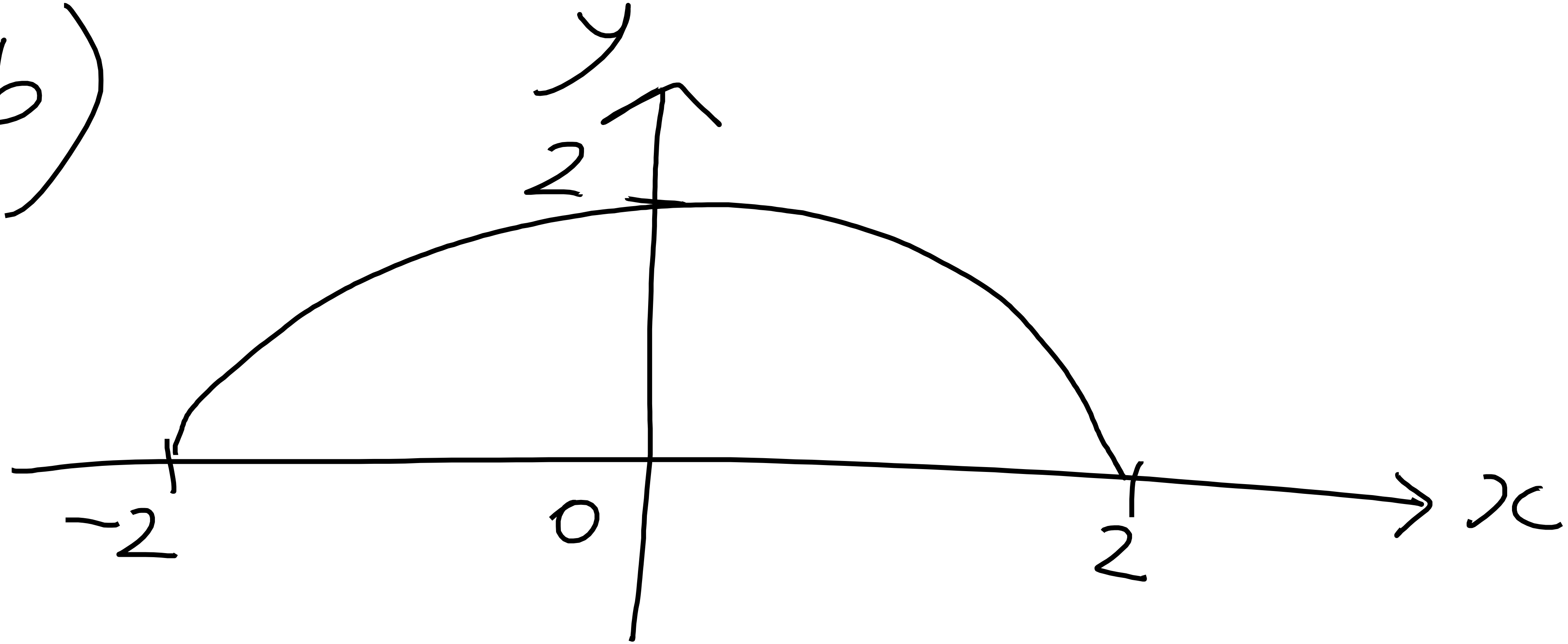
$$\bar{y} = \frac{1}{m} \iint_R y \, \rho(x, y) \, dA$$

$$= \int_0^1 \int_{y-1}^{1-y} y \, dx \, dy$$

$$= \int_0^1 y(2-2y) \, dy$$

$$= \frac{1}{3} \quad \#$$

3b)



$\rho(x, y) = 1 + \frac{y}{2}$ (indep. of x)

$\Rightarrow$ left & right equally dense

$\Rightarrow \bar{x} = 0$

$$m = \iint_R 1 + \frac{y}{2} \, dA$$

$$= \int_0^{\pi} \int_0^{2} \left(1 + \frac{1}{2} r \sin\theta\right) r \, dr \, d\theta$$

$$= \cdots = \frac{6\pi + 8}{3}$$

#

$$\bar{x} = \frac{1}{m}\iint_R x\,\rho(x,y)\,dA$$

$$= \frac{3}{6\pi+8}\int_0^{\pi}\int_0^{2} (r\cos\theta)\left(1+\tfrac{1}{2}r\sin\theta\right) \cdot r\,dr\,d\theta$$

$= \cdots$ $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$= 0$ $\quad \int \sin 2\theta \, d\theta = \dfrac{-\cos 2\theta}{2}$

$$\bar{y} = \frac{1}{m} \iint_R y\left(1+\frac{y}{2}\right) dA$$

$$= \frac{3}{6\pi+8} \int_0^{\pi} \int_0^{2} (r\sin\theta)\left(1+\tfrac{1}{2}r\sin\theta\right) r\,dr\,d\theta$$

$$= \dots$$

$$= \frac{3}{6\pi+8} \int_0^{\pi} \left(\frac{8}{3}\sin\theta + \underline{\underline{2\sin^2\theta}}\right) d\theta$$

Recall

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\boxed{2\sin^2\theta = 1 - \cos 2\theta}$$

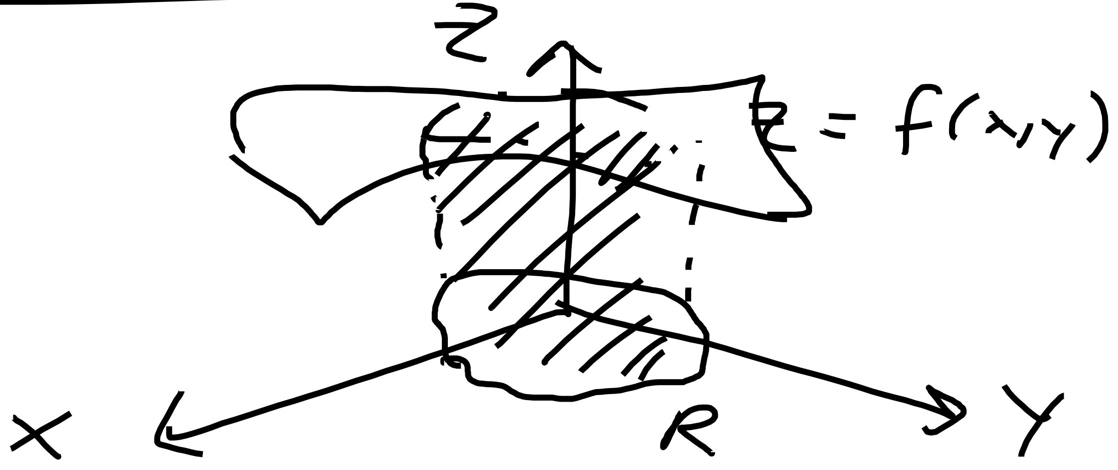
$= \ldots$

$$= \frac{3\pi + 16}{6\pi + 8} \quad \#$$

Basic Qn 7

z needs to be always positive

$$\boxed{\text{Volume under surface } z = f(x,y) = \iint_R z \, dA}$$



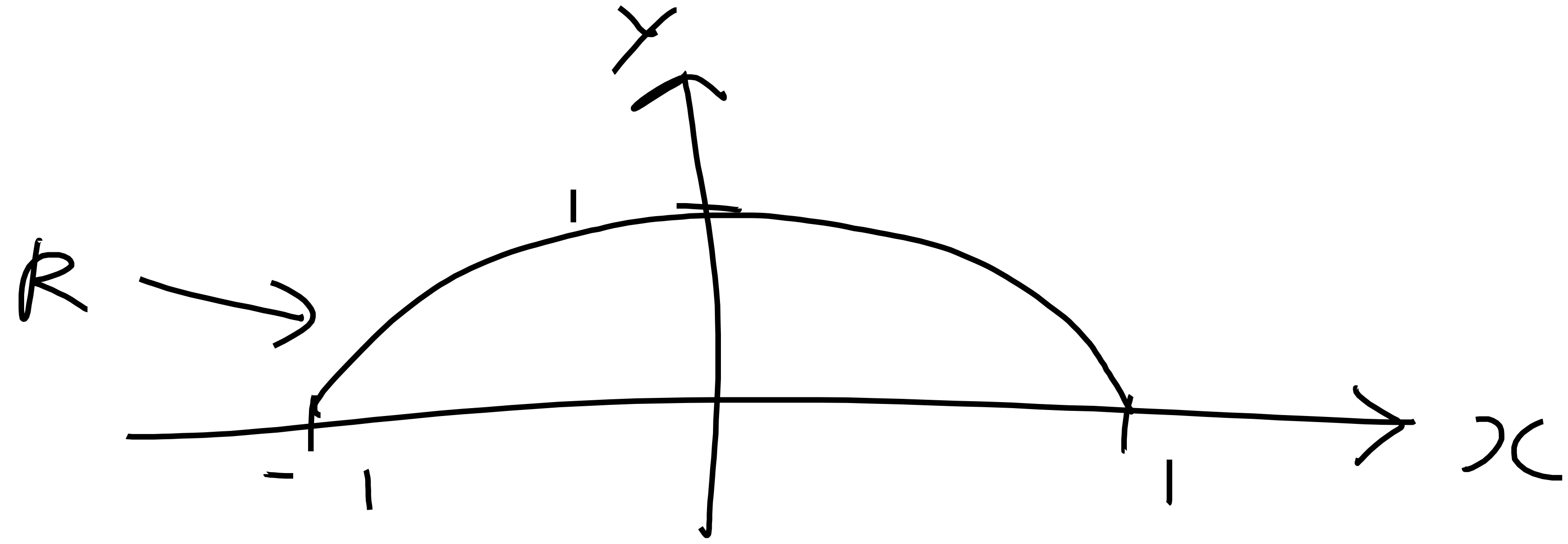
$$V = \iint_R z \, dA$$

$$= \iint_R \underbrace{5 - \sqrt{1 + x^2 + y^2}}_{\text{always positive}} \, dA$$

$-1 \le x \le 1$

$-1 \le y \le 1$

$$= \int_0^{\pi} \int_0^{1} \left(5 - \sqrt{1 + r^2}\right) r \, dr \, d\theta$$



$$= \int_0^{\pi} \int_0^1 5r - r\sqrt{1+r^2} \, dr \, d\theta$$

$$= \int_0^{\pi} \left[\frac{5r^2}{2} - \frac{1}{3}(1+r^2)^{3/2} \right]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{5}{2} - \frac{1}{3}(2)^{3/2} + \frac{1}{3} \, d\theta$$

$$= \ldots = \frac{(17-4\sqrt{2})\pi}{6}$$

#

Side working

$$\frac{1}{2}\,\frac{u^{3/2}}{3/2}$$

$$\int r\sqrt{1+r^2}\,dr = \frac{1}{2}\int (u)^{\frac{1}{2}}\,du$$

Let $u = 1 + r^2$

or $u = r^2$

$$\frac{du}{dr} = 2r$$

$$du = 2r\,dr \implies r\,dr = \frac{1}{2}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2}$$

$$= \frac{1}{3} u^{3/2}$$

$$= \frac{1}{3}(1+r^2)^{3/2}$$