MA1511

Tutorial 1 (B2)

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- Discussion Problems (ALL)
- Basic Qns (some)
- In Class Assignment (25 min)

Q1. (i) Use $\boxed{e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k}$

to show $\frac{1}{x^x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x \ln x)^k$.

Step 1) Replace x by $-x \ln x$

$$e^{-x \ln x} = \sum_{k=0}^{\infty} \frac{1}{k!} (-x \ln x)^k$$

Fact: ① $e^{\ln t} = t$; $n \ln x = \ln x^n$

$$e^{\ln x^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x \ln x)^k$$

$$\frac{1}{x^x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (x \ln x)^k$$

(shown).

1 (iii) Integrate both sides.

$$\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(x\ln x)^k\,dx$$

Interchange Integral & Summation (Valid for MA1511)

$$= \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k}{k!}(x\ln x)^k\,dx$$

Const w.r.t. x

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 (x\ln x)^k\,dx$$

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{k=0}^{\infty} \frac{\cancel{(-1)^k}}{\cancel{k!}} \boxed{\cancel{(-1)^k}\,\frac{\cancel{k!}}{(k+1)^{k+1}}}$$

$$\boxed{\begin{aligned}(-1)^k(-1)^k &= (-1)^{2k}\\ &= (1)^k\\ &= 1\end{aligned}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}}$$

Replace k by $k-1$

$$= \sum_{k-1=0}^{\infty} \frac{1}{(k-1+1)^{k-1+1}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^k} \qquad \#$$

Q2) $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\boxed{(S_\infty \text{ for GP}) \quad \begin{aligned} a &= 1 \\ r &= x \end{aligned}}$$

Show

$$\sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$$

Replace x by $\underline{-x^2}$

$$\sum_{k=0}^{\infty}(-x^2)^k = \frac{1}{1+x^2}$$

$$\boxed{\begin{gathered}(\underset{-1}{-}x^2)^k = (-1)^k(x^{2k}) \\ (ab)^k = a^k b^k\end{gathered}}$$

$$\sum_{k=0}^{\infty}(-1)^k x^{2k} = \frac{1}{1+x^2} \quad \#$$

Fact: $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$

$$\int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx = \int \frac{1}{1+x^2}\,dx$$

$$\sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx = \tan^{-1}x + C$$

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{2k+1} = \tan^{-1}x + C$$

Let $x=0 \Rightarrow 0 = 0 + C$

Q3) Geometric Progression

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

a : first term $= 1.05X$

r : common ratio $= 1.05$

n : number of terms $= 20$

$X is deposited every year for 20 years.

1st $X earns 20 years interest.

$$a_1 = 1.05^{20} X$$

2nd $X earns 19 years interest

$$a_2 = 1.05^{19} X$$

. . . . . .

$$a_{20} = 1.05^1 X$$ first term

$$\text{Total} = 1.05^1 X + 1.05^2 X + \cdots + 1.05^{20} X$$

$$= \frac{1.05X(1.05^{20}-1)}{1.05-1} \geq 200000$$

$$X \geq 5760.5$$

Ans 5761

Q4) When does

$$\sum_{k=1}^{\infty} \left(\frac{x-1}{2x-3} \right)^k$$

converge?

(finite answer)

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{a}{1-r}$$

$$= \frac{1/2}{1-\frac{1}{2}} = 1$$

$$\left| \frac{x-1}{2x-3} \right| < 1$$

← Solve this

$$\left|\frac{x-1}{2x-3}\right| < 1$$

$$\left|\frac{x-1}{2x-3}\right| - 1 < 0$$

eg.

$|a| < 2$

$-2 < a < 2$

$|a| > 2$

$a > 2$ or $a < -2$

① Square both side

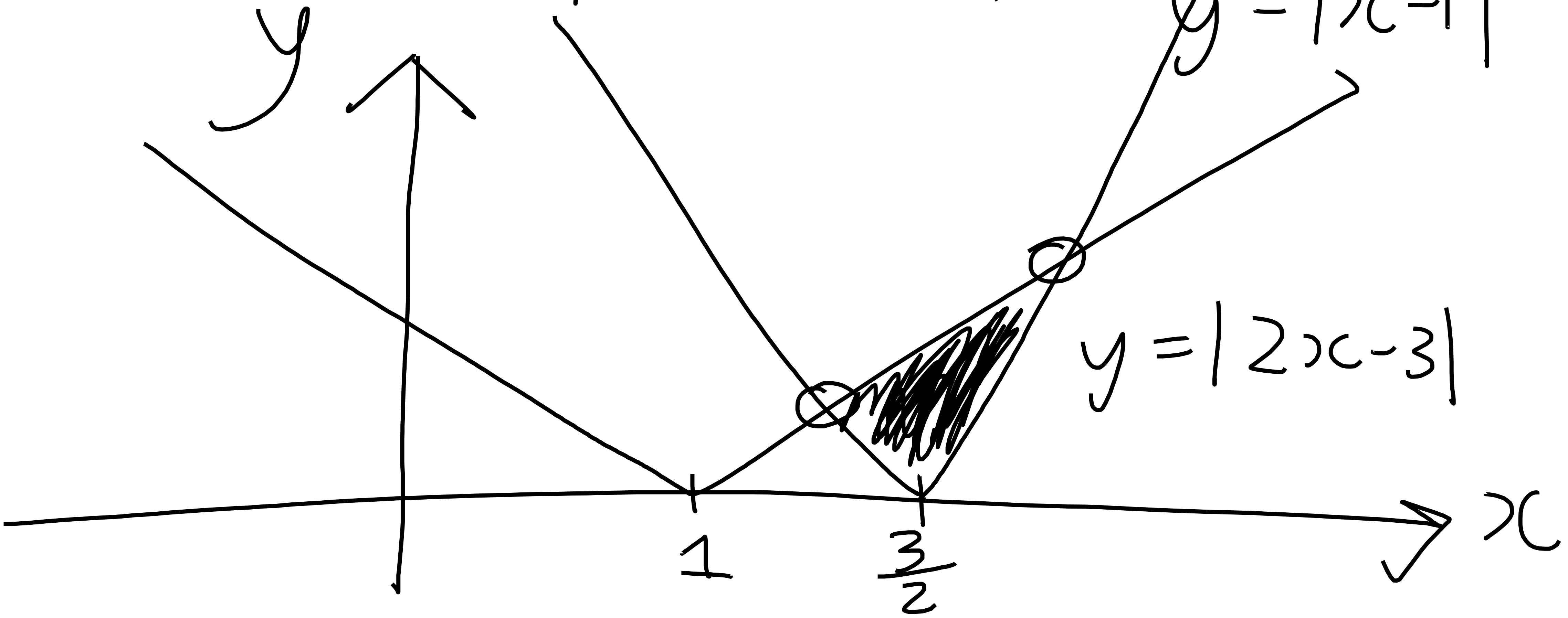
$$\left(\frac{x-1}{2x-3}\right)^2 < 1$$

Correct but tedious

(lead to extra answer that you need reject manually)

② Graphical Method

$$|x-1| < |2x-3|$$



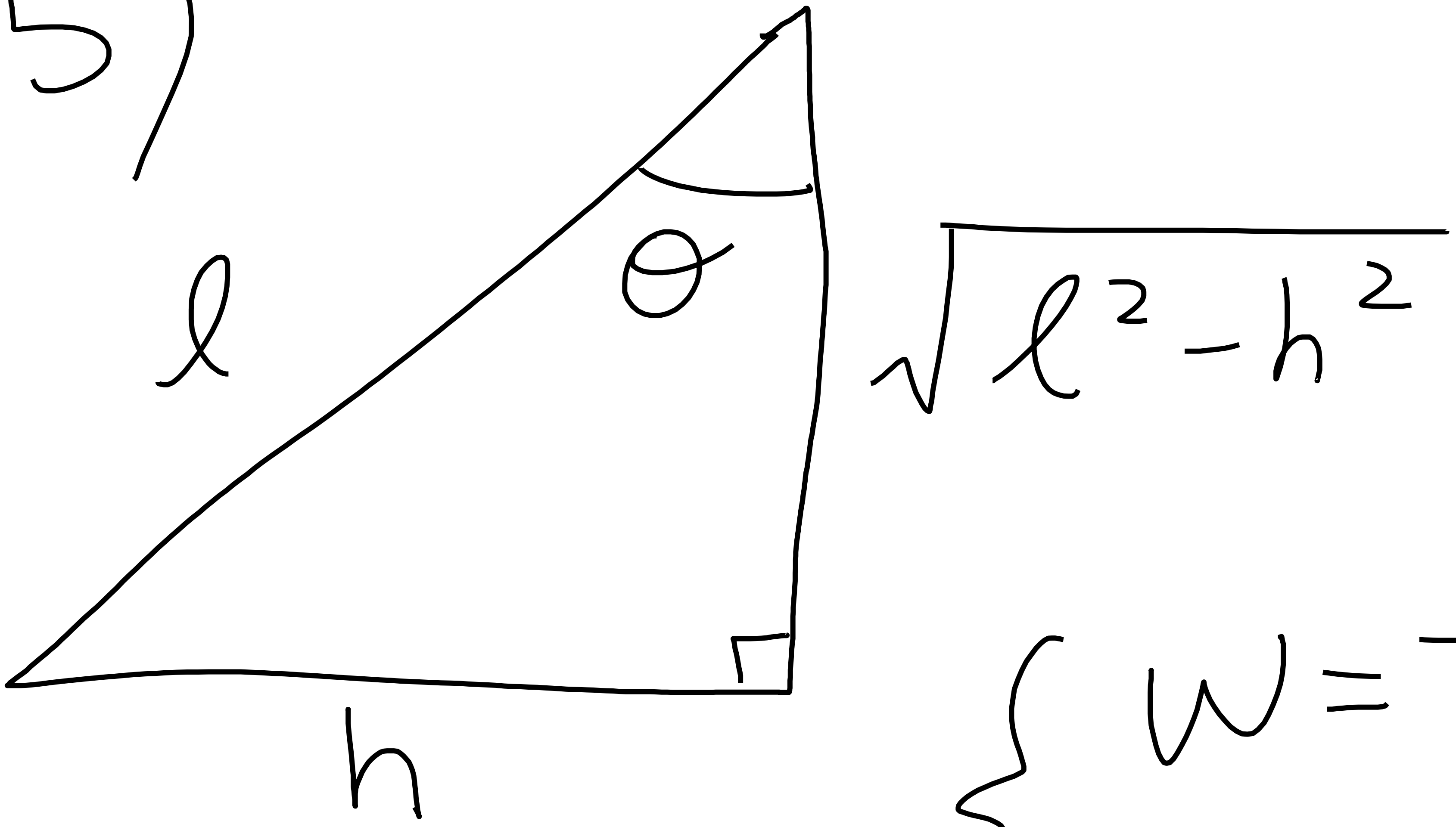
$x-1=3-2x$	$x-1=2x-3$
$x=\frac{4}{3}$	$x=2$

Ans :

$x<\frac{4}{3}$ or

$x>2$ #

Q5)



$$\begin{cases} W = T\cos\theta \\ F = T\sin\theta \end{cases}$$

Show : $F = \dfrac{Wh}{\sqrt{\ell^2 - h^2}}$

no T , no θ

$\dfrac{F}{W} = \dfrac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} = \tan\theta$

$\tan\theta = \dfrac{h}{\sqrt{\ell^2 - h^2}}$

$$F = W\tan\theta$$

$$= \frac{Wh}{\sqrt{l^2 - h^2}} \quad \# \text{ (shown)}$$

Binomial:

$$(1+x)^{\alpha} = \left(1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots\right)$$

need to be 1

$$\frac{Wh}{\sqrt{l^2-h^2}} = \frac{Wh}{\sqrt{l^2}\sqrt{1-\frac{h^2}{l^2}}}$$

$$= \frac{Wh}{l}\left(1-\underbrace{\frac{h^2}{l^2}}_{\text{small}}\right)^{-\frac{1}{2}}$$

small ✓ $\frac{h}{l}$ small ✓

$\frac{h^2}{l^2}$ small ✓

$$= \frac{Wh}{l}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{h^2}{l^2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{h^2}{l^2}\right)^2+\dots\right]$$

$$\approx \frac{wh}{l}\left[1+\frac{h^2}{2l^2}+\frac{3h^4}{8l^4}\right]$$

(shown)

Q7c) $\sum_{k=1}^{\infty} \frac{(-1)^k (2x+3)^k}{k!}$

Let $\boxed{a_k = \frac{(-1)^k (2x+3)^k}{k!}}$

Ratio test / Root test

✓ (due to factorial)

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1}(2x+3)^{k+1}}{(k+1)!} \times \frac{k!}{(-1)^k (2x+3)^k}\right|$$

multiply by reciprocal

$$= \left|\frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{(2x+3)^{k+1}}{(2x+3)^k} \cdot \frac{k!}{(k+1)!}\right|$$

$$= \frac{|2x+3|}{(k+1)} \to 0 \text{ as } k \to \infty$$

for all x

$0 < 1$

By ratio test, it converges for all real x.

$\therefore$ Radius of convergence $= \infty$ ✓