T1Q5 Supplementary Solutions

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Note: For D.E. two solutions may look very different, but end up being the same. So if your answer is different, it doesn't necessarily mean you are wrong. Two ways to check your answer are by Graphmatica, and by differentiating your answer to see if you get back the original question.

1 Q5a

Let $u = 1 + y + 2x$. Then $y = u - 1 - 2x$, so $y' = u' - 2$.

Substitute these into the question, we get $u' - 2 = \frac{-2u+3}{u}$.

So

$$\frac{du}{dx} = \frac{-2u+3}{u} + \frac{2u}{u} = \frac{3}{u}.$$

Separating variables, we get $\int u \, du = \int 3 \, dx$.

Integrating, we get $\frac{u^2}{2} = 3x + c$.

Substitute back $u = 1 + y + 2x$, we get

Answer: $\frac{1}{2}(1 + y + 2x)^2 = 3x + c$.

1.1 Alternative Method (see official solution)

Let $v = 2x + y$ instead. You should get

Answer: $(2x + y) + \frac{1}{2}(2x + y)^2 = 3x + c$.

2 Q5b

Let $u = x + y + 1$. Rearrange and differentiate, then $y' = u' - 1$.

Substitute the above into the question to get $\frac{du}{dx} - 1 = (\frac{u}{u+2})^2$, thus

$$\frac{du}{dx} = \frac{u^2 + (u+2)^2}{(u+2)^2}.$$

$$\int \frac{(u+2)^2}{u^2 + (u+2)^2}\, du = \int 1\, dx$$

$$\int \frac{u^2 + 4u + 4}{2u^2 + 4u + 4}\, du = x + c$$

$$\int \frac{u^2 + 4u + 4}{u^2 + 2u + 2}\, du = 2x + c$$

(multiply both sides by 2 to simplify the denominator)

$$\int 1 + \frac{2u + 2}{u^2 + 2u + 2}\, du = 2x + c$$

$$u + \ln |u^2 + 2u + 2| = 2x + c$$

$x + y + 1 + \ln |x^2 + y^2 + 1 + 2xy + 2x + 2y + 2x + 2y + 2 + 2| = 2x + c$
(substitute back $u = x + y + 1$)

Answer: $x + y + 1 + \ln |(x + y)^2 + 4x + 4y + 5| = 2x + c.$

2.1 Alternative Method (see official solution)

Let $v = x + y$ instead. You should get

Answer: $x + y + \ln |(x + y)^2 + 4x + 4y + 5| = 2x + c.$

3 Q5c

Make y' the subject to get

$$y' = \frac{x+y+1}{x-y+3}. \tag{†}$$

(At this point, note that letting $u = x + y + 1$ or $u = x - y + 3$ **doesn't work**.)

We need to use this new technique to further simplify the expression.

Key step: Let

$$x = X + \alpha$$
$$y = Y + \beta.$$

Then

$$x + y + 1 = X + Y + (\alpha + \beta + 1)$$
$$x - y + 3 = X - Y + (\alpha - \beta + 3).$$

We want to choose appropriate α, β such that

$$\alpha + \beta + 1 = 0$$
$$\alpha - \beta + 3 = 0$$

Why are we doing this? The reason is to further simplify the expression, so that the resulting expression is purely an expression in X and Y, without constants. So that we can divide throughout by X to make the expression **a function of** $\frac{Y}{X}$.

Solve the above simultaneous equation to get $\boxed{\alpha = -2}$ and $\boxed{\beta = 1}$.

Proposition 3.1. We prove that

$$\frac{dY}{dX} = \frac{dy}{dx}.$$

Proof. Since $x = X - 2$, so $\frac{dx}{dX} = 1$. Since $Y = y - 1$, so $\frac{dY}{dy} = 1$. By chain rule,

$$\frac{dY}{dX} = \frac{dY}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dX} = (1) \cdot \frac{dy}{dx} \cdot (1) = \frac{dy}{dx}.$$

□

Substitute the previous information into our very first expression (†) to get

$$\frac{dY}{dX} = \frac{X + Y}{X - Y} = \frac{1 + (\frac{Y}{X})}{1 - (\frac{Y}{X})}.$$

Let $V = \frac{Y}{X}$, so $Y = XV$.

By Product Rule,

$$\frac{dY}{dX} = V + X\frac{dV}{dX}.$$

$$\frac{1 + V}{1 - V} = V + X\frac{dV}{dX}$$

$$X\frac{dV}{dX} = \frac{1 + V}{1 - V} - \frac{V - V^2}{1 - V} = \frac{V^2 + 1}{1 - V}$$

$$\int \frac{1 - V}{V^2 + 1}\, dV = \int \frac{1}{X}\, dX$$

Using the formula $\int \frac{1}{x^2+1}\, dx = \tan^{-1} x$,

$$\tan^{-1}(V) - \frac{1}{2}\ln|V^2 + 1| = \ln|X| + c.$$

Substitute back $V = \frac{y-1}{x+2}$, $X = x + 2$ to get

Answer:

$$\tan^{-1}(\frac{y - 1}{x + 2}) - \frac{1}{2}\ln\left((\frac{y - 1}{x + 2})^2 + 1\right) = \ln|x + 2| + c.$$