Second Order Linear D.E. Summary

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1 Homogenous D.E.

$y'' + ay' + by = 0.$

Solve the **Characteristic Equation**: $\lambda^2 + a\lambda + b = 0.$

Case 1) Two real roots $\lambda_1, \lambda_2 \implies \boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$

Case 2) Real double root $\lambda \implies \boxed{y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$

Case 3) Complex Conjugate roots $\lambda_1, \lambda_2 = -\frac{a}{2} \pm iw$**, where** $w = \sqrt{b - \frac{a^2}{4}}$

$\implies \boxed{y = e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx)}$

2 Non-homogenous D.E.

General solution of non-homogenous D.E.:

$$y = y_h + y_p,$$

where y_h is the general solution of the homogenous equation, and y_p is the particular solution (with no arbitrary constants).

2.1 Method of Undetermined Coefficients (Guess and try method)

$y'' + p(x)y' + q(x)y = r(x)$.

Only works if $r(x)$ is polynomial, exponential, sine or cosine (or sum/product of these).

Polynomial: Try y=Polynomial (e.g. $y = Ax^2 + Bx + C$ or $y = Bx + C$.)

Exponential (e^{kx}): Try $y = ue^{kx}$, where u is a function of x.

Trigonometric ($\sin kx$ or $\cos kx$): Convert to complex differential equation. Try $z = ue^{ikx}$. Then take real/imaginary part of z for cosine/sine respectively.

2.2 Method of variation of parameters

$y'' + p(x)y' + q(x)y = r(x)$.

Step 1) Solve the homogenous D.E. $y'' + p(x)y' + q(x)y = 0$.

Get solution of the form $y_h = c_1y_1 + c_2y_2$.

Step 2) Let

$$u = -\int \frac{y_2 r}{W}\,dx$$

and

$$v = \int \frac{y_1 r}{W}\,dx$$

where W is the Wronskian

$$W = y_1y_2' - y_1'y_2.$$

Particular solution: $y_p = uy_1 + vy_2$.

General solution: $y = y_h + y_p$.