# PDE Summary

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In this document, we summarize the chapter on PDE (Partial Differential Equations), with applications to exam questions.

## Contents

# 1 Separation of Variables

1. Let $u(x, y) = X(x)Y(y)$.
2. Rearrange the equation such that LHS is a function of $x$ only, RHS is a function of $y$ only.
3. Thus, LHS=RHS=some constant $k$.
4. Solve the two separate ODEs.

## 1.1 2016 Q8(a)

Solve $xu_x + u_y = 0$, $x > 0$, given that $u(x, 0) = 2x$.

Outline of solution:

1. Let $u(x, y) = X(x)Y(y)$. Then $u_x = X'Y$, $u_y = XY'$.
2. So $xX'Y + XY' = 0$. Divide by $XY$ throughout, we get $\frac{xX'}{X} + \frac{Y'}{Y} = 0$
3. $\implies \frac{xX'}{X} = -\frac{Y'}{Y} = k$.
4. Solve the two ODEs: $\frac{X'}{X} = \frac{k}{x}$ and $-\frac{Y'}{Y} = k$.

# 2 Wave Equation

$$c^2 y_{xx} = y_{tt},$$

where $y(t, 0) = y(t, \pi) = 0$, $y(0, x) = f(x)$, $y_t(0, x) = 0$.

## 2.1 Solution of Wave Equation (with Fourier sine coefficients)

$$y(t, x) = \sum_{n=1}^{\infty} b_n \sin(nx) \cos(nct)$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx.$$

### 2.2 d'Alembert's solution

$y(t,x) = \frac{1}{2}[f(x+ct) + f(x-ct)]$.

### 2.3 2016 8(b)

$c = 3$, so $y(t,x) = \sum_{n=1}^\infty b_n \sin(nx)\cos(3nt)$.

Thus $y(0,x) = \sum_{n=1}^\infty b_n \sin(nx) = 2\sin x + 3\sin 6x$.

By comparing coefficients, $b_1 = 2$, $b_6 = 3$, $b_n = 0$ for all $n \neq 1, 6$.

Thus $y(t,x) = 2\sin x\cos 3t + 3\sin 6x\cos 18t$.

## 3 Heat Equation

$$u_t = c^2 u_{xx},$$

$u(0,t) = u(L,t) = 0$, $u(x,0) = f(x)$.

### 3.1 Solution of Heat Equation

$$u(x,t) = \sum_{n=1}^\infty b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{\pi^2 n^2 c^2}{L^2}t\right),$$

where $b_n$ are Fourier sine coefficients of $f(x)$ (same formula as that in wave equation).

### 3.2 2015 Q8(b)

Similar idea as 2016 Q8(b).

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = 12\sin 9x - 7\sin 4x$$

and compare coefficients to find the $b_n$. Refer to official solution for more details.

# 4 Laplace Equation

$u_{xx} + u_{yy} = 0$.

## 4.1 Case 1) Nonzero on Top of square

$u(x,0) = u(0,y) = u(\pi,y) = 0$ and $u(x,\pi) = f(x)$.

### 4.1.1 Solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(ny)$$

where

$$c_n = \frac{\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}{\sinh(n\pi)}.$$

## 4.2 Case 2) Nonzero on Bottom of square

$u(x,0) = f(x)$, $u(0,y) = u(\pi,y) = u(x,\pi) = 0$.

### 4.2.1 Solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh[n(\pi - y)]$$

where $c_n$ is the same formula as before.