Laplace Transform

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In this document, we list some of the basic concepts of Laplace Transform, and tips on when to use each equation.

1 Laplace Transform

1.1 Laplace transform of f

$F(s) = L(f) = \int_0^\infty e^{-st} f(t)\, dt$

Tip: Use this equation when the questions contains the words "show from the definition". E.g. Tutorial 7 Q4(i).

1.2 Inverse transform of $F(s)$

$f(t) = L^{-1}(F(s))$

1.3 Linearity

$$L(af(t) + bg(t)) = aL(f) + bL(g)$$
$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F) + bL^{-1}(g)$$

2 List of common Laplace Transforms

$$L(e^{at}) = \frac{1}{s-a} \qquad (s > a)$$
$$L(1) = \frac{1}{s} \qquad (s > 0)$$
$$L(\cos wt) = \frac{s}{s^2 + w^2}$$
$$L(\sin wt) = \frac{w}{s^2 + w^2}$$
$$L(t^n) = \frac{n!}{s^{n+1}}$$
$$L(f') = sL(f) - f(0) \qquad (s > a)$$
$$L(f'') = s^2 L(f) - sf(0) - f'(0)$$
$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$
$$L\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s} L(f) \qquad (s > 0, s > a)$$

3 s-shifting

If $L(f) = F(s)$, $s > a$, then

$$\boxed{L(e^{ct} f(t)) = F(s-c)},$$

$s - c > a$.

Tip: Use this when doing Laplace Transform of a function with an exponential factor e^{ct}. E.g. Tutorial 7 Q1(a). Note that the reverse direction can sometimes be used as well: $L^{-1}[F(s-c)] = e^{ct}f(t)$.

4 t-shifting

If $L(f(t)) = F(s)$, then

$$\boxed{L(f(t-a)u(t-a)) = e^{-as}F(s)}$$

Tip: Frequently, we use the reverse direction

$$L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a).$$

E.g. Tutorial 7 Q2(b).

5 Delta function

$\delta(t)$: infinitely tall and narrow spike at $t = 0$.

$\delta(t-a)$: infinitely tall and narrow spike at $t = a$.

$$\boxed{L[\delta(t-a)] = e^{-as}}$$

5.1 Two properties of delta function

$$\int_0^\infty \delta(t-a)\,dt = 1$$

$$\int_0^\infty \delta(t-a)g(t)\,dt = g(a)$$

for $a \geq 0$.

Tip: Use delta function when the keywords "suddenly", "burst", etc. appear. E.g. Tutorial 8 Q2, Tutorial 7 Q5.

6 Unit step function

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases}$$

For $0 < a < b$,

$$u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b. \end{cases}$$

Tip: Use unit step function for questions that require a force to "switch on / switch off" at certain times. E.g. Tutorial 7 Q6.

$$\boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$