For Tutorial 8 Q2, once we reach the integral

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2 - y^2}\, dx\, dy,$$

it is tempting to use polar coordinates "$y = r\sin\theta$" (incorrect) to convert the integral into

$$\text{Incorrect: } \int_0^{\pi/2} \int_0^r \sqrt{r^2 - r^2\sin^2\theta}\; r\, dr\, d\theta = \frac{r^3}{3}.$$

This is incorrect because the r in the question is a **constant**, while for polar coordinates, the radius "r" should be a **variable**.

Hence if we really want to use polar coordinates, we need to **call our radius something else, say** R. So $y = R\sin\theta$ instead.

It is doable, but it turns out it is quite tedious:

$$\begin{aligned}
V_1 &= \int_0^{\pi/2} \int_0^r \sqrt{r^2 - R^2\sin^2\theta}\; R\, dR\, d\theta \\
&= \int_0^{\pi/2} \frac{-1}{2\sin^2\theta} \left[\int_0^r \sqrt{r^2 - R^2\sin^2\theta}(-2R\sin^2\theta)\, dR \right] d\theta \\
&= \int_0^{\pi/2} \frac{-1}{2\sin^2\theta} \left[\frac{(r^2 - R^2\sin^2\theta)^{3/2}}{3/2} \right]_{R=0}^{R=r} d\theta \\
&= \int_0^{\pi/2} \frac{-1}{2\sin^2\theta} \left[\frac{r^3\cos^3\theta - r^3}{3/2} \right] d\theta \\
&= \frac{1}{3} r^3 \int_0^{\pi/2} \frac{1 - \cos^3\theta}{\sin^2\theta}\, d\theta \\
&= \frac{2}{3} r^3 \quad \text{(see explanation below for why the integral is 2)}
\end{aligned}$$

There are two ways to see why

$$\int_0^{\pi/2} \frac{1 - \cos^3\theta}{\sin^2\theta}\, d\theta = 2.$$

First way is to use your Graphical Calculator's numerical integration program. Second way is to use this method (improper integral):

$$\begin{aligned}
\int_0^{\pi/2} \frac{1-\cos^3\theta}{\sin^2\theta}\,d\theta &= \lim_{a\to0^+}\int_a^{\pi/2} \frac{1-\cos^3\theta}{\sin^2\theta}\,d\theta \\
&= \lim_{a\to0^+}\int_a^{\pi/2} \csc^2\theta - \frac{(1-\sin^2\theta)}{\sin^2\theta}\cos\theta\,d\theta \\
&= \lim_{a\to0^+}\left[-\cot\theta + \frac{1}{\sin\theta} + \sin\theta\right]_a^{\pi/2} \\
&= \lim_{a\to0^+}\left[2 - (-\cot a + \frac{1}{\sin a} + \sin a)\right] \\
&= 2 - \lim_{a\to0^+}(\frac{-\cos a + 1 + \sin^2 a}{\sin a}) \\
&= 2 - \lim_{a\to0^+}\frac{\sin a + 2\sin a\cos a}{\cos a} \quad \text{(L'Hopital)} \\
&= 2.
\end{aligned}$$