Introduction

Tutorial 5 Q7 is a very famous mathematical problem known as the "Basel Problem", solved by Euler in 1734. Basically, it asks for the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Three hundred years ago, this was considered a very hard problem and even famous mathematicians of the time like Leibniz, De Moivre, and the Bernoullis could not solve it.

Euler showed (using another method different from ours) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

bringing him great fame among the mathematical community. It is a beautiful equation; it is surprising that the constant π, usually related to circles, appears here.

1 Squaring the Fourier sine series

Assume that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Then squaring this series formally,

$$\begin{aligned}(f(x))^2 &= (\sum_{n=1}^{\infty} b_n \sin nx)^2 \\ &= \sum_{n=1}^{\infty} b_n^2 \sin^2 nx + \sum_{n \neq m} b_n b_m \sin nx \sin mx.\end{aligned}$$

To see why the above hold, see the following concrete example:

$$\begin{aligned}(a_1 + a_2 + a_3)^2 &= (a_1^2 + a_2^2 + a_3^2) + (a_1a_2 + a_1a_3 + a_2a_1 + a_2a_3 + a_3a_1 + a_3a_2) \\ &= \sum_{n=1}^{3} a_n^2 + \sum_{n \neq m} a_n a_m.\end{aligned}$$

2 Integrate term by term

We assume that term by term integration is valid.

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n^2 \sin^2 nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n\neq m} b_n b_m \sin nx \sin mx\,dx.$$

Recall in your notes (pg 9) that

$$\int_{-\pi}^{\pi}\sin nx \sin mx\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}.$$

So

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n^2 \sin^2 nx\,dx &= \frac{1}{\pi}\sum_{n=1}^{\infty} b_n^2 (\int_{-\pi}^{\pi}\sin^2 nx\,dx) \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} b_n^2(\pi) \\ &= \sum_{n=1}^{\infty}(b_n)^2.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n\neq m} b_n b_m \sin nx \sin mx\,dx &= \frac{1}{\pi}\sum_{n\neq m} b_n b_m (\int_{-\pi}^{\pi}\sin nx \sin mx\,dx) \\ &= \frac{1}{\pi}\sum_{n\neq m} b_n b_m(0) \\ &= 0.\end{aligned}$$

So

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \sum_{n=1}^{\infty}(b_n)^2.$$

3 Apply Parseval's Identity to $f(x) = x$

By Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}(\frac{2(-1)^{n+1}}{n})^2.$$

Simplifying, we get

$$\frac{1}{\pi}\cdot\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \sum_{n=1}^{\infty}\frac{4}{n^2}.$$

$$\frac{1}{\pi}(\frac{2\pi^3}{3}) = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$