1 $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ (L'Hopital's Rule Proof)

This limit appears in Tutorial 3 Q1e, and is a useful and interesting result to know. Note especially that the method "$\lim_{n\to\infty}(1+\frac{1}{n})^n = 1^\infty = 1$" is **incorrect**.

Proof. We will prove $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ instead, and this implies

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e.$$

First, we will find the limit $\lim_{x\to\infty}\ln(1+\frac{1}{x})^x$.

$$\begin{aligned}
\lim_{x\to\infty}\ln(1+\frac{1}{x})^x &= \lim_{x\to\infty} x\ln(1+\frac{1}{x}) \quad \text{(Bringing down the power } x\text{)} \\
&= \lim_{x\to\infty}\frac{\ln(1+x^{-1})}{x^{-1}} \\
&= \lim_{x\to\infty}\frac{\frac{1}{1+x^{-1}}(-x^{-2})}{-x^{-2}} \quad \text{(L'Hopital's Rule)} \\
&= \lim_{x\to\infty}\frac{1}{1+x^{-1}} \\
&= 1.
\end{aligned}$$

So $\lim_{x\to\infty}(1+\frac{1}{x})^x = e^1$. □

2 Exercise

If you are interested, you can try to prove

$$\lim_{n\to\infty}(1+\frac{c}{n})^n = e^c$$

where $c \in \mathbb{R}$.